

Generalised vielbeins and non-linear realisations

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Abstract

We briefly review why the non-linear realisation of the semi-direct product of a group with one of its representations leads to a field theory defined on a generalised space-time equipped with a generalised vielbein. We give formulae, which only involve matrix multiplication, for the generalised vielbein, the Cartan forms and their transformations. We consider the generalised space-time introduced in 2003 in the context of the non-linear realisation of the semi-direct product of E_{11} and its first fundamental representation. For this latter theory we give explicit expressions for the generalised vielbein up to and including the levels associated with the dual graviton in four, five and eleven dimensions and for the IIB theory in ten dimensions. We also compute the generalised vielbein, up to the analogous level, for the non-linear realisation of the semi-direct product of very extended $SL(2)$ with its first fundamental representation, which is a theory associated with gravity in four dimensions.

1. Introduction

We begin by giving a very brief review of the general theory of non-linear realisations. While some aspects of this are very well known, the non-linear realisations that involve a group whose generators are associated with space-time are less well known. In particular we will make it clear why the non-linear realisations which lead to space-times automatically encode a generalised vielbein.

A non-linear realisation of a group G with local subgroup H is constructed from a group element $g \in G$ which is subject to the transformations

$$g \rightarrow g_0 g, \quad \text{for } g_0 \in G \quad \text{and also} \quad g \rightarrow gh, \quad \text{for } h \in H \quad (1.1)$$

where g_0 is a rigid transformation and h a local transformation. The meaning of rigid and local will be discussed below. The non-linear realisation is an action, or set of equations of motion, that are invariant under the transformations of equation (1.1). The dynamics is usually constructed from the Cartan forms \mathcal{V} which are inert under the rigid g_0 transformations but transform under the h transformations as

$$\mathcal{V} \rightarrow \mathcal{V}' = h^{-1} \mathcal{V} h + h^{-1} dh \quad (1.2)$$

Before explaining the particular type of non-linear realisation that will be discussed in this paper it will be instructive to briefly discuss the three types of non-linear realisation.

1.1 Type 1

Non-linear realisations were first introduced to understand the scattering of pions and it was through this work that it became understood that symmetry was to play a crucial role in particle physics [1]. The theoretical underpinning of this method was set out in the classic papers of reference [2]. This work involved a group G which contained generators that were internal, that is, not associated with space-time. Space-time was introduced in an adhoc manner by taking the group element g , and so the parameters it contains, to depend on the chosen space-time, which in the application at that time, was just four dimensional Minkowski space-time. As a result, the parameters of the group element g became the fields of the theory defined on the chosen space-time. The rigid transformation g_0 is a constant group element, while the local transformation h is taken to depend on the chosen space-time and so can be used to gauge away parts of g . The Cartan forms can be written as

$$\mathcal{V} = g^{-1} dg = P_I T^I + Q_i H^i \quad (1.3)$$

where H^i are the generators of H and T^I the remaining generators of G . When the group is such that the commutators between generators of H with T^I lead again to the generators T^I (the reductive case), the forms $P \equiv P_I T^I$ transform homogeneously and can be used to construct an invariant action which is just the space-time integral of $\text{Tr} P^2$. The dynamics of the pions were found to be very well described, in the limit of small pion mass, by the non-linear realisation of $SU(2) \otimes SU(2)$ with respect to its diagonal subgroup $SU(2)$.

1.2 Type 2

A non-linear realisation at the other extreme is one where all the generators of the group G are associated with "space-time". In this case we have a simple coset space, often written as $\frac{G}{H}$, which has been studied for a very long time, at least in the mathematics literature. In this case the H transformations enforce the usual equivalence relation that ensures that the group elements of G are regarded as equivalent if they belong to the same coset. Modulo this relation the parameters in the group element label the points in the coset, which for the application that physicists have in mind are the points in space-time. Thus in this case we have no fields.

For these non-linear realisations the Cartan forms can be written as

$$\mathcal{V} = g^{-1}dg = dx^\Pi E_\Pi^A l_A + dx^\Pi \omega_{\Pi i} H^i \quad (1.4)$$

where H^i are the generators of H and l_A the remaining generators of G . The objects E_Π^A define a preferred basis of one forms $dx^\Pi E_\Pi^A$ at every point of the coset which are just those swept out out, using the natural action of the group on the coset, from a basis of one forms at the origin of the coset. As a result we can interpret the objects E_Π^A as vielbeins on the coset space. The objects $\omega_{\Pi i}$ can be thought of as the spin-connection on the coset. One can easily verify, using equation (1.1) that both of these objects transform as vielbeins and spin connections should on their A and i indices respectively under the local H transformations.

The classic example of such a non-linear realisation is to take G to be the Poincare group, which can be written as the semi-direct product of the Lorentz group, $SO(1, D-1)$, with a set of generators in its vector representation, denoted $l^{SO(1, D-1)}$, with the local subgroup H being the Lorentz group. We denote this semi-direct product by $SO(1, D-1) \otimes_s l^{SO(1, D-1)}$. Another example is superspace which is the non-linear realisation of the super Poincare group with the local subgroup being the Lorentz group [3].

1.1 Type 3

The final type of non-linear realisation is built from a group G that has some generators that are associated with space-time and some that are not. For simplicity, and as it is the case we wish to consider in this paper, we will take the group to be of a semi-direct product structure, that is, of the form $G = G_1 \otimes_s l_1$ where G_1 is a Lie group and l_1 is one of its representations. We denote the generators of G_1 by R^α and those of the l_1 representation by l_A . The Lie algebra for this group can be written in the form

$$[R^\alpha, R^\beta] = f^{\alpha\beta}_{\underline{\delta}} R^\delta \quad (1.5)$$

and

$$[R^\alpha, l_A] = -(D^\alpha)_A{}^B l_B \quad (1.6)$$

The Jacobi identity, implies that the generators l_A belong to representation of G_1 and so the matrices in the above equation obey the matrix identity

$$[D^\alpha, D^\beta] = f^{\alpha\beta}_{\underline{\delta}} D^\delta \quad (1.7)$$

The commutators of the l_A generators must be consistent with the Jacobi identities and we will take them, for simplicity, to commute.

The group element $g \in G$ is constructed from the generators l_A and R^α and can be written in the form

$$g = g_l g_A \equiv e^{x^A l_A} e^{A_\alpha(x) R^\alpha} \quad (1.8)$$

The parameters of the l_A generators can be interpreted as the coordinates of a generalised space-time while the parameters of the R^α generators are taken to depend on the coordinates of space-time and are fields defined on the generalised space-time. Rigid in this case means that the group element g_0 does not depend on the generalised space-time and so its parameters are constants. The local subalgebra H of G used in the non-linear realisation is a subalgebra of G_1 and local H transformations have a group element h which does depend on space-time. This transformation can be used to gauge away some of the fields in g . However, once this has been done we have to carry out compensating H transformation to preserve the form of the group element g under a rigid g_0 transformation. This last type of non-linear realisation is a hybrid of the first two types; if we take no l_1 generators then it is of type one while if we take no generators of the type R^α then it is of type two.

The Cartan forms belong to the Lie algebra of G and so can be written as

$$\mathcal{V} = \mathcal{V}_l + \mathcal{V}_A \quad (1.9)$$

where \mathcal{V}_l contains the generators of l_1 and \mathcal{V}_A the generators of G_1 and as such we can write them in the form

$$\mathcal{V}_l \equiv dx^\Pi E_\Pi{}^A l_A = g_A^{-1} dx^\Pi l_{\Pi} g_A, \quad \text{and} \quad \mathcal{V}_A \equiv dx^\Pi G_{\Pi, \underline{\alpha}} R^\alpha = g_A^{-1} dg_A \quad (1.10)$$

We can interpret the objects $E_\Pi{}^A$ as the vielbein on the generalised space-time.

One of the early examples of this type of non-linear realisation was to take $G = GL(D) \otimes_s l^{GL(D)}$ where $l^{GL(D)}$ is the vector representation of $SL(D)$, or equivalently its first fundamental representation [4,5]. This non-linear realisation gives, with a judicious choice of a few undetermined constants, Einstein's theory of gravity [4,5]. A more recent example, and the one of interest to us here, is to take G to be the semi-direct product of E_{11} and its first fundamental representation l_1 , denoted $E_{11} \otimes_s l_1$ [6]. This is a special case of non-linear realisations constructed from the groups $G = G^{+++} \otimes_s l_1$ where G^{+++} is the very extension of any finite dimensional semi-simple Lie algebra and l_1 is the first fundamental representation of G^{+++} . We note that $E_{11} = E_8^{+++}$. The non-linear realisations $A_{D-3}^{+++} \otimes_s l_1$ [6] and $D_{D-2}^{+++} \otimes_s l_1$ [8] are conjectured to be the low energy effective actions for gravity and the closed bosonic string in D dimensions respectively. A more detailed review of non-linear realisation can be found in [9].

In this paper we will consider non linear realisation of the last type that is the non-linear realisation of $G = G_1 \otimes_s l_1$. In section two we derive expressions for the generalised vielbein, Cartan forms and their transformations that require no more than matrix multiplication. In section three we consider the non-linear realisation $E_{11} \otimes_s l_1$ and compute the generalised vielbeins in eleven, five and four dimensions and the IIB theory in ten dimensions up to levels three, four, two and five respectively. In section four we give the initial steps in the construction of the non-linear realisation of the $A_1^{+++} \otimes_s l_1$ and compute the generalised vielbein up to level two. This later theory is conjectured to be the complete

low energy effective action for four dimensional gravity. In appendix A we recall, up to the level associated with the dual graviton, the $E_{11} \otimes_s l_1$ algebra in the decompositions appropriate to eleven and four dimensions and for five dimensions and the IIB theory in ten dimensions we give these algebras for the first time.

2 Formulae for the generalised vielbein and Cartan forms

In this section we consider the non-linear realisation of the semi-direct product of a group G_1 with one of its representations l_1 which we denote by $G_1 \otimes_s l_1$ and so we are discussing the case of type three of the section one. In this section the l_1 representation can be any representation and not just the first fundamental representation. We will take the generators of the l_1 representation to commute. It is straightforward to modify the discussion to the case when the generators of the l_1 representation do not commute, but form a group.

The generators of the group G_1 in the non-linear realisation are usually taken to be abstract objects, but if we take them to be in the l_1 representation then it is straightforward to derive expressions, that involve no more than matrix multiplication, for the generalised vielbein, their transformations, and the Cartan forms. These are well known for the non-linear realisation of $GL(D) \otimes_s l^{GL(D)}$ [4] and were recently given [10] for the generalised vielbein for $E_{11} \otimes_s l_1$.

The generalised vielbein is defined in equation (1.10) and it is straightforward to evaluate using equation (1.6) to find that it is given by

$$E_{\Pi}{}^A = (e^A)_{\Pi}{}^A \quad (2.1)$$

where $(\mathcal{A})_{\Pi}{}^A \equiv A_{\underline{\alpha}}(D^{\underline{\alpha}})_{\Pi}{}^A$ and the expression on the right-hand side is evaluated by expanding the exponential and using matrix multiplication. Taking the generators of the G_1 algebra to be in the l_1 representations in the expression for the Cartan forms of equation (1.10) we find that

$$\begin{aligned} -[\mathcal{V}_A, l_A] &= dx^{\Pi} G_{\Pi, \underline{\alpha}}(D^{\underline{\alpha}})_A{}^B l_B = -[g_A^{-1} dg_A, l_A] \\ &= -g_A^{-1} d(g_A l_A g_A^{-1}) g_A = (E^{-1})_A{}^{\Delta} dE_{\Delta}{}^B l_B \end{aligned} \quad (2.2)$$

and so

$$G_{\Pi, A}{}^B \equiv G_{\Pi, \underline{\alpha}}(D^{\underline{\alpha}})_A{}^B = (E^{-1})_A{}^{\Delta} \partial_{\Pi} E_{\Delta}{}^B \quad (2.3)$$

Using the expression for the vielbein of equation (2.1) we find that

$$G_{\Pi, A}{}^B = \left(\frac{(1 - e^{-A})}{A} \wedge \partial_{\Pi} \mathcal{A} \right)_A{}^B = \left(\partial_{\Pi} \mathcal{A} - \frac{1}{2} [\mathcal{A}, \partial_{\Pi} \mathcal{A}] + \frac{1}{3!} [\mathcal{A}, [\mathcal{A}, \partial_{\Pi} \mathcal{A}]] + \dots \right)_A{}^B \quad (2.4)$$

where we have used the identity

$$e^{-D} de^D = \frac{(1 - e^{-D})}{D} \wedge dD \quad (2.5)$$

valid for any operator, or matrix, D and where $D^n \wedge dD \equiv [D, [D, [D, \dots [D, dD]]] \dots]$.

The action of the rigid transformation $g_0 \in G^{+++}$, which can be written in the form $g_0 = e^{a_{\underline{\beta}} R^{\underline{\beta}}}$, can also be given in explicit form. As the generators l_A form a representation of G^{+++} under this transformation, equation (1.1) implies that

$$g_l \rightarrow g'_l = g_0 g_l g_0^{-1}, \quad \text{and} \quad g_A \rightarrow g'_A \rightarrow g_0 g_A \quad (2.6)$$

Using equation (1.6) the first equation is found to imply the coordinate change

$$x^{\Delta} \rightarrow x^{\Delta'} = x^{\Pi} (e^{-a \cdot D})_{\Pi}{}^{\Delta} \quad (2.7)$$

where $(a \cdot D)_{\Pi}{}^{\Delta} = a_{\underline{\beta}} (D^{\underline{\beta}})_{\Pi}{}^{\Delta}$. While the change in the vielbein can be found by considering

$$(g_A^{-1})' l_{\Pi} g'_A = (g_A^{-1}) g_0^{-1} l_{\Pi} g_0 g_A = (e^{a \cdot D})_{\Pi}{}^{\Delta} (g_A^{-1})_{\Delta} l_{\Delta} g_A = (e^{a \cdot D})_{\Pi}{}^{\Delta} E_{\Delta}{}^C l_C \quad (2.8)$$

and as a result

$$E_{\Pi}{}^A \rightarrow E_{\Pi}{}^{A'} = (e^{a \cdot D})_{\Pi}{}^{\Delta} E_{\Delta}{}^A, \quad \text{or equivalently in matrix notation} \quad e^{A'} = e^{a \cdot D} e^A \quad (2.9)$$

We note that $dx^{\Pi} E_{\Pi}{}^A$ is inert under rigid g_0 transformations as it should be.

It is often useful not to parameterise the group element g_A by a single exponential but by a product of exponentials. In this case one just replaces the above matrix expressions by the corresponding products, for example, if let set $g_A = e^{A_1 \cdot R} \dots e^{A_n \cdot R}$ then the vielbein takes the form.

$$E_{\Pi}{}^A = (e^{\mathcal{A}_1} \dots e^{\mathcal{A}_n})_{\Pi}{}^A \quad (2.10)$$

where $\mathcal{A}_1 = A_1 \cdot (D)_{\Pi}{}^A$ and there are analogous expressions for the above formulae.

To proceed further we will need the Cartan Involution I_c which can be taken to act on the generators of E_{11} as $I_c(R^{\alpha}) = -R^{-\alpha}$. In fact we have in previous papers taken a plus sign for some of the involutions, but this can be undone by redefining the negative generators. The Cartan involution acts on the l_1 representation to give another representation denoted by \bar{l}^A as $I_c(l_A) = -J_{AB}^{-1} \bar{l}^B$ for a suitable matrix J_{AB} . Acting on the commutator of equation (1.6) with the Cartan involution we find that

$$[R^{\alpha}, \bar{l}^A] = \bar{l}^B (\bar{D}^{\alpha})_B{}^A \quad (2.11)$$

where

$$(\bar{D}^{\alpha})_B{}^A = (J D^{-\alpha} J^{-1})^A{}_B, \quad \text{or in matrix form} \quad \bar{D}^{\alpha} = (J D^{-\alpha} J^{-1})^T \quad (2.12)$$

For the case of $E_{11} \otimes_s l_1$, the l_1 representation is a lowest weight representation with lowest weight state P_1 while \bar{l}_1 is a highest weight representation with highest weight state \bar{P}^1 where P_a , $a = 1, \dots, D$ are the usual space-time translation generators and \bar{P}^a , $a = 1, \dots, D$.

We take the local subalgebra in the $G_1 \otimes l_1$ non-linear realisation to be the Cartan involution subgroup of G_1 which consists of group elements which obey $I_c(h) = h$. Following similar arguments one finds that the local $h = e^{b_{\underline{\alpha}}(R^{\underline{\alpha}} - R^{-\underline{\alpha}})}$ transformation of the generalised vielbein is given by

$$E_{\Pi}{}^A \rightarrow E_{\Pi}{}^{A'} = e^{A'} = (e^{\mathcal{A}} e^{b_{\underline{\beta}}(D^{\underline{\beta}} - D^{-\underline{\beta}})})_{\Pi}{}^A = E_{\Pi}{}^B (e^{b_{\underline{\beta}}(D^{\underline{\beta}} - D^{-\underline{\beta}})})_B{}^A \quad (2.11)$$

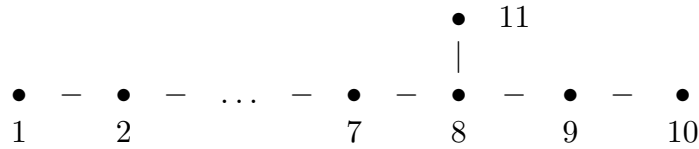
It is sometimes useful to construct the dynamics not from the Cartan forms, but from the object $M = g_A I_c(g_A^{-1})$ which transforms as $M \rightarrow M' = g_0 M I_c(g_0^{-1})$. We note that $I_c(M^{-1}) = M$ and so M can be written in the form $M = e^{\phi_{\underline{\alpha}}(R^{\underline{\alpha}} + R^{-\underline{\alpha}})}$ which confirms that M is a group element that belongs to the coset. The matrix representation of M is given by

$$\begin{aligned} M_A{}^B l_B &\equiv M^{-1} l_A M = (e^{\phi_{\underline{\alpha}}(D^{\underline{\alpha}} + D^{-\underline{\alpha}})})_A{}^B l_B = I_c(g) g^{-1} l_A g I_c(g^{-1}) \\ &= (e^{\mathcal{A}})_A{}^B I_c(g) l_A I_c(g^{-1}) = (e^{\mathcal{A}})_A{}^B J_{BC}^{-1} I_c(g) \bar{l}^C g^{-1} = (e^{\mathcal{A}} e^{\tilde{\mathcal{A}}})_A{}^B l_B \end{aligned} \quad (2.13)$$

where the matrix $\tilde{\mathcal{A}} = A_{\underline{\alpha}} D^{-\underline{\alpha}}$. The transformation of M can be written, in matrix form, as $M \rightarrow M' = e^{a_{\underline{\alpha}} D^{\underline{\alpha}}} M e^{a_{\underline{\alpha}} D^{-\underline{\alpha}}}$

3 Explicit computation of the E_{11} generalised vielbein at low levels

In this section we will consider the non-linear realisation of $E_{11} \otimes_s l_1$ with the local subgroup being the Cartan involution invariant subalgebra of E_{11} ; the analogue of the maximal compact subalgebra. This non-linear realisation has been conjectured to be the low energy effective action describing strings and branes [6,11]. The representations of E_{11} can be studied by decomposing them into representations of a finite-dimensional Lie algebras, obtained by removing one node from the Dynkin diagram of E_{11} . The Dynkin diagram of E_{11} is given by



The theories with different number of space-time dimensions emerge when computes the non-linear realisation of the $E_{11} \otimes_s l_1$ algebra when decomposed into the algebras that follow by removing the different possible nodes [12-14]. In this paper we are interested in four particular cases: removing node 11 leads to $GL(11)$ algebra that corresponds to 11-dimensional theory, removing node 9 results in 10-dimensional type IIB theory with $GL(10) \times SL(2, R)$ algebra, removing node 5 leads to $GL(5) \times E_6$ algebra that describes 5-dimensional theory, and, finally, removing node 4 leads to $GL(4) \times E_7$ algebra that corresponds to the 4-dimensional theory. The fields and coordinates in D dimensions can be classified by a level that is given by the number of down minus up $SL(D)$ indices except that one adds one for the coordinates and divides the results by three in eleven dimensions and two for the ten dimensional IIB theory.

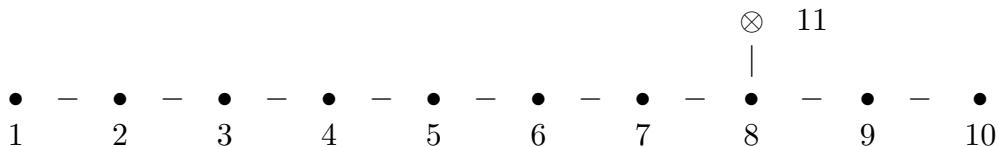
The l_1 representation decomposed in the way suitable to D dimensions leads to a generalised space-time that contains at level zero the usual coordinates x^a and at level one coordinates that are scalars under the Lorentz group but transform as the 10, $\overline{16}$, $\overline{27}$, 56 and $248 \oplus 1$ of $SL(5)$, $SO(5,5)$, E_6 , E_7 and E_8 for D equal to seven, six, five, four and three dimensions respectively [16,17]. The corresponding generalised vielbeins have been partially constructed at low levels for these generalised space-times using the $E_{11} \otimes_s l_1$ non-linear realisation. One of the first examples was the construction of the generalised vielbein for the five dimensional theory up to level two [15] which, in conjunction with the corresponding generalised space-time, was used to find all maximally supersymmetric gauged supergravities. In the four dimensional theory reference [18] computed the 56 by 56 vielbein that arises in the space of the level one coordinates [18]. The full generalised vielbein up to and including level one in the four dimensional theory was given in [19]. The generalised vielbein, but restricted to the space of the level one coordinates, was also subsequently computed in [20] in dimensions four up to seven inclusive. The eleven dimensional generalised vielbein was computed up to level two in [21]. A metric that appeared in the duality invariant first quantised actions studied in reference [26] was used in reference [25] to discuss theories formulated on a seven dimensional space-time. However, we note that this generalised space-time is just the part of l_1 representation of E_{11} at level one in seven dimensions [6,16,17] and the vielbein, or equivalently the metric, is a truncation of the vielbeins found earlier in the context of E_{11} papers.

Siegel theory [22], sometimes called doubled field theory, was developed in 1993. This was motivated by string theory and it consists of a theory with the same massless fields as appear in the NS-NS sector of the superstring, but defined in a 20-dimensional space-time that transformed in the vector representation of $O(10,10)$. A generalised vielbein defined on this space-time, was found in reference [22], it played an important part in the construction of Siegel theory. The Virasoro operators appeared in construction and they were to contain a corresponding metric which agreed with that found when reducing string theory on a torus. This theory was subsequently formulated as the non-linear realisation of $E_{11} \otimes_s l_1$ in ten dimensions at level zero [23]. The extension of this theory to include the R-R sector is just the level one contribution and it was first found in reference [24]. The generalised vielbein computed from this later viewpoint agrees with that found earlier.

In this section we calculate the generalised vielbein in eleven, five and four dimensions and also the for the ten dimensional IIB theory at much higher levels using the $E_{11} \otimes_s l_1$ non-linear realisation.

3.1 $D = 11$

The eleven dimensional theory is obtained by deleting node 11 from the Dynkin diagram of E_{11} .



and decomposing the $E_{11} \otimes_s l_1$ into representations of $GL(11)$ [11]. In this section we will restrict ourselves with level 3 calculations. The non-negative level generators of the E_{11}

are

$$K^a{}_b; R^{a_1 a_2 a_3}; R^{a_1 \dots a_6}; R^{a_1 \dots a_8, b}. \quad (3.1.1)$$

The negative level generators are

$$R_{a_1 a_2 a_3}; R_{a_1 \dots a_6}; R_{a_1 \dots a_8, b}. \quad (3.1.2)$$

The l_1 representation contains the generators [6]

$$P_a; Z^{a_1 a_2}; Z^{a_1 \dots a_5}; Z^{a_1 \dots a_8}, Z^{a_1 \dots a_7, b}. \quad (3.1.3)$$

The group element $g = g_l g_A$ can be parametrised in the following way:

$$\begin{aligned} g_l &= \exp \left(x^a P_a + x_{a_1 a_2} Z^{a_1 a_2} + x_{a_1 \dots a_5} Z^{a_1 \dots a_5} + x_{a_1 \dots a_8} Z^{a_1 \dots a_8} + x_{a_1 \dots a_7, b} Z^{a_1 \dots a_7, b} \right), \\ g_A &= \exp \left(h_a{}^b K^a{}_b \right) \exp \left(A_{a_1 \dots a_8, b} R^{a_1 \dots a_8, b} \right) \exp \left(A_{a_1 \dots a_6} R^{a_1 \dots a_6} \right) \exp \left(A_{a_1 a_2 a_3} R^{a_1 a_2 a_3} \right), \end{aligned} \quad (3.1.4)$$

where we have introduced the generalised coordinates [6]

$$x^a; x_{a_1 a_2}; x_{a_1 \dots a_5}; x_{a_1 \dots a_8}, x_{a_1 \dots a_7, b}. \quad (3.1.5)$$

We have used the local subalgebra to gauge away part of the g_A group element and we have the, by now well known, fields of the $E_{11} \otimes_s l_1$ non-linear realisation up to level three, namely, the graviton, the three and six form gauge fields and the dual graviton [11]. The corresponding generalised tangent space structure is obvious and the tangent space group is $I_c(E_{11})$ which at lowest level is just the Lorentz group and at higher levels has an algebra can be found in reference [6] and also the book of reference [8].

The generalised vielbein is defined in equation (1.10) and, while one can straightforwardly compute it using the commutators of appendix A.1, we will find it using the matrix expression of equation (2.10), which in eleven dimensions takes the form

$$E_{\Pi}{}^A = e^{A_0} e^{A_3} e^{A_2} e^{A_1} \quad (3.1.6)$$

where

$$\mathcal{A}_0 \equiv h_a{}^b D_a{}^b, \mathcal{A}_1 \equiv A_{a_1 a_2 a_3} D^{a_1 a_2 a_3}, \mathcal{A}_2 \equiv A_{a_1 \dots a_6} D^{a_1 \dots a_6}, \mathcal{A}_3 \equiv A_{a_1 \dots a_8, b} D^{a_1 \dots a_8, b} \quad (3.1.7)$$

We begin with the level zero matrix which is given by the expression

$$\begin{aligned} dx \cdot (\mathcal{A}_0) \cdot l &= -[h_a{}^b K^a{}_b, dx^c P_c + dx_{c_1 c_2} Z^{c_1 c_2} \\ &+ dx_{c_1 \dots c_5} Z^{c_1 \dots c_5} + dx_{c_1 \dots c_8} Z^{c_1 \dots c_8} + dx_{c_1 \dots c_7, c} Z^{c_1 \dots c_7, c}] \end{aligned} \quad (3.1.8)$$

from which we conclude that

$$(\mathcal{A}_0) = \begin{pmatrix} h_a{}^b & 0 & 0 & 0 & 0 \\ 0 & -2\delta_{[a_1}^{[b_1} h_{a_2]}^{b_2]} & 0 & 0 & 0 \\ 0 & 0 & -5\delta_{[a_1 \dots a_4}^{[b_1 \dots b_4} h_{a_5]}^{b_5]} & 0 & 0 \\ 0 & 0 & 0 & -8\delta_{[a_1 \dots a_7}^{[b_1 \dots b_7} h_{a_8]}^{b_8]} & 0 \\ 0 & 0 & 0 & 0 & k_{a_1 \dots a_7, b}^{c_1 \dots c_7, d} \end{pmatrix} - \frac{1}{2} h_c{}^c I \quad (3.1.9)$$

where $k_{a_1 \dots a_7, b}^{c_1 \dots c_7, d} = -7\delta_b^a \delta_{[a_1 \dots a_6}^{[b_1 \dots b_6} h_{a_7]}^{b_7]} - \delta_{b_1 \dots b_7}^{a_1 \dots a_7} h_b^a + 8\delta_{[b_1 \dots b_7}^{[a_1 \dots a_7} h_b]^{a]}$ and I is the identity matrix.

It then follows that

$$e^{\mathcal{A}_0} = (\det e)^{-\frac{1}{2}} \begin{pmatrix} e_\mu^a & 0 & 0 & 0 & 0 \\ 0 & (e^{-1})_{a_1 a_2}^{\mu_1 \mu_2} & 0 & 0 & 0 \\ 0 & 0 & (e^{-1})_{a_1 \dots a_5}^{\mu_1 \dots \mu_5} & 0 & 0 \\ 0 & 0 & 0 & (e^{-1})_{a_1 \dots a_8}^{\mu_1 \dots \mu_8} & 0 \\ 0 & 0 & 0 & 0 & (e^{-1})_{a_1 \dots a_7, b}^{\mu_1 \dots \mu_7, \nu} \end{pmatrix}, \quad (3.1.10)$$

where $e_\mu^b = (e^h)_\mu^b$ and

$$(e^{-1})_{a_1 \dots a_n}^{\mu_1 \dots \mu_n} = (e^{-1})_{[a_1}^{\mu_1} \dots (e^{-1})_{a_n]}^{\mu_n},$$

$$(e^{-1})_{a_1 \dots a_7, b}^{\mu_1 \dots \mu_7, \nu} = (e^{-1})_{[a_1}^{\mu_1} \dots (e^{-1})_{a_7]}^{\mu_7} (e^{-1})_b^\nu - (e^{-1})_{[a_1}^{\mu_1} \dots (e^{-1})_{a_7]}^{\mu_7} (e^{-1})_{b]}^\nu. \quad (3.1.11)$$

We now compute \mathcal{A}_1 in a similar way by considering

$$dx \cdot (\mathcal{A}_1) \cdot l = -[A_{a_1 a_2 a_3} R^{a_1 a_2 a_3}, dx^c P_c + x_{c_1 c_2} Z^{c_1 c_2} + dx_{c_1 \dots c_5} Z^{c_1 \dots c_5} + dx_{c_1 \dots c_8} Z^{c_1 \dots c_8} + dx_{c_1 \dots c_7, c} Z^{c_1 \dots c_7, c}] \quad (3.1.12)$$

from which we conclude, using the commutators of appendix A.1, that

$$(\mathcal{A}_1) = \begin{pmatrix} 0 & -3A_{ab_1 b_2} & 0 & 0 & 0 \\ 0 & 0 & -\delta_{[b_1 b_2}^{a_1 a_2} A_{b_3 b_4 b_5]} & 0 & 0 \\ 0 & 0 & 0 & -\delta_{[b_1 \dots b_5}^{a_1 \dots a_5} A_{b_6 b_7 b_8]} & k_{b_1 \dots b_5 b_6 b_7, b}^{a_1 \dots a_5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.1.13)$$

where $k_{b_1 \dots b_5 b_6 b_7, b}^{a_1 \dots a_5} = -\delta_{[b_1 \dots b_5}^{a_1 \dots a_5} A_{b_6 b_7] b} + \delta_{[b_1 \dots b_5}^{a_1 \dots a_5} A_{b_6 b_7 b]}$. Proceeding in a similar way we find that

$$(\mathcal{A}_2) = \begin{pmatrix} 0 & 0 & 3A_{ab_1 \dots b_5} & 0 & 0 \\ 0 & 0 & 0 & \delta_{[b_1 b_2}^{a_1 a_2} A_{b_3 \dots b_8]} & \delta_{[b_1 b_2}^{a_1 a_2} A_{b_3 \dots b_7] b} - \delta_{[b_1 b_2}^{a_1 a_2} A_{b_3 \dots b_7 b]} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.1.14)$$

and

$$(\mathcal{A}_3) = \begin{pmatrix} 0 & 0 & 0 & \frac{3}{2} A_{b_1 \dots b_8, a} & -\frac{4}{3} A_{a[b_1 \dots b_7], b} + \frac{4}{3} A_{a[b_1 \dots b_7, b]} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.1.15)$$

To compute the generalised vielbein we just need to evaluate the matrix expression of equation (3.1.6), being careful to evaluate the unusual index sets, we find that

$$E_{\Pi}^A = (\det e)^{-\frac{1}{2}}$$

$$\begin{pmatrix} e_{\mu}^a & e_{\mu}^b \alpha_{b|a_1 a_2} & e_{\mu}^b \alpha_{b|a_1 \dots a_5} & e_{\mu}^b \alpha_{b|a_1 \dots a_8} & e_{\mu}^b \alpha_{b|a_1 \dots a_7, a} \\ 0 & (e^{-1})_{a_1 a_2}^{\mu_1 \mu_2} & (e^{-1})_{b_1 b_2}^{\mu_1 \mu_2} \beta_{a_1 \dots a_5}^{b_1 b_2} & (e^{-1})_{b_1 b_2}^{\mu_1 \mu_2} \beta_{a_1 \dots a_8}^{b_1 b_2} & (e^{-1})_{b_1 b_2}^{\mu_1 \mu_2} \beta_{a_1 \dots a_7, b}^{b_1 b_2} \\ 0 & 0 & (e^{-1})_{a_1 \dots a_5}^{\mu_1 \dots \mu_5} & (e^{-1})_{b_1 \dots b_5}^{\mu_1 \dots \mu_5} \gamma_{a_1 \dots a_8}^{b_1 \dots b_5} & (e^{-1})_{b_1 \dots b_5}^{\mu_1 \dots \mu_5} \gamma_{a_1 \dots a_7, b}^{b_1 \dots b_5} \\ 0 & 0 & 0 & (e^{-1})_{a_1 \dots a_8}^{\mu_1 \dots \mu_8} & 0 \\ 0 & 0 & 0 & 0 & (e^{-1})_{a_1 \dots a_7, b}^{\mu_1 \dots \mu_7, \nu} \end{pmatrix}, \quad (3.1.16)$$

where the symbols in the first line of this matrix are given by

$$\begin{aligned} \alpha_{a|a_1 a_2} &= -3 A_{aa_1 a_2}, \quad \alpha_{a|a_1 \dots a_5} = 3 A_{aa_1 \dots a_5} + \frac{3}{2} A_{a[a_1 a_2} A_{a_3 a_4 a_5]}, \\ \alpha_{a|a_1 \dots a_8} &= \frac{3}{2} A_{a_1 \dots a_8, a} - 3 A_{a[a_1 \dots a_5} A_{a_6 a_7 a_8]}, \\ \alpha_{a|a_1 \dots a_7, b} &= \frac{4}{3} A_{a[a_1 \dots a_7, b]} + 3 A_{a[a_1 \dots a_5} A_{a_6 a_7 b]} - \frac{4}{3} A_{a[a_1 \dots a_7], b} \\ &\quad - 3 A_{a[a_1 \dots a_5} A_{a_6 a_7] b} - \frac{1}{2} A_{a[a_1 a_2} A_{a_3 a_4 a_5} A_{a_6 a_7] b}, \end{aligned} \quad (3.1.17)$$

the symbols in the second line are given by

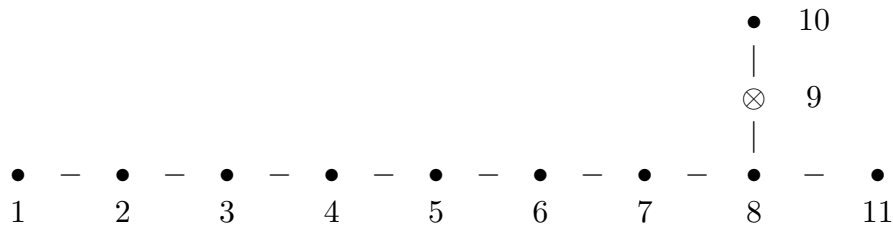
$$\begin{aligned} \beta_{a_1 \dots a_5}^{b_1 b_2} &= -\delta_{[a_1 a_2}^{b_1 b_2} A_{a_3 a_4 a_5]}, \quad \beta_{a_1 \dots a_8}^{b_1 b_2} = \delta_{[a_1 a_2}^{b_1 b_2} A_{a_3 \dots a_8]}, \\ \beta_{a_1 \dots a_7, b}^{b_1 b_2} &= \delta_{[a_1 a_2}^{b_1 b_2} A_{a_3 \dots a_7] b} + \frac{1}{2} \delta_{[a_1 a_2}^{b_1 b_2} A_{a_3 a_4 a_5} A_{a_6 a_7] b} - \delta_{[a_1 a_2}^{b_1 b_2} A_{a_3 \dots a_7] b}, \end{aligned} \quad (3.1.18)$$

and, finally, the symbols in the third line are given by

$$\gamma_{a_1 \dots a_8}^{b_1 \dots b_5} = -\delta_{[a_1 \dots a_5}^{b_1 \dots b_5} A_{a_6 a_7 a_8]}, \quad \gamma_{a_1 \dots a_7, b}^{b_1 \dots b_5} = \delta_{[a_1 \dots a_5}^{b_1 \dots b_5} A_{a_6 a_7 b]} - \delta_{[a_1 \dots a_5}^{b_1 \dots b_5} A_{a_6 a_7] b}. \quad (3.1.19)$$

3.2 $D = 10$

There are two ways of obtaining a ten-dimensional theory: removing node 10 leads to type IIA supergravity theory, while removing node 9 leads to type IIB supergravity [12]. We will be interested in the latter. The corresponding Dynkin diagram is



In this case all the generators fall into representations of $GL(10) \times SL(2, R)$. The non-negative level generators of adjoint representation up to level 5 are [12,27]

$$\begin{aligned} K^a{}_b, R_{\alpha\beta}; R_{\alpha}^{a_1 a_2}; R^{a_1 \dots a_4}; R_{\alpha}^{a_1 \dots a_6}; R_{\alpha\beta}^{a_1 \dots a_8}; K^{a_1 \dots a_7, b}, \\ R_{\alpha\beta\gamma}^{a_1 \dots a_{10}}, R_{\alpha}^{a_1 \dots a_8, b_1 b_2}, R_{\alpha}^{a_1 \dots a_9, b}. \end{aligned} \quad (3.2.1)$$

The negative level generators are

$$\begin{aligned} R_{a_1 a_2}^{\alpha}; R_{a_1 \dots a_4}; R_{a_1 \dots a_6}^{\alpha}; R_{a_1 \dots a_8}^{\alpha\beta}; K_{a_1 \dots a_7, b} \\ R_{a_1 \dots a_{10}}^{\alpha\beta\gamma}, R_{a_1 \dots a_8, b_1 b_2}^{\alpha}, R_{a_1 \dots a_9, b}^{\alpha}. \end{aligned} \quad (3.2.2)$$

The l_1 representation generators up to level five are

$$\begin{aligned} P_a; Z_{\alpha}^a; Z^{a_1 a_2 a_3}; Z_{\alpha}^{a_1 \dots a_5}; Z_{\alpha\beta}^{a_1 \dots a_7}; Z^{a_1 \dots a_7}, Z^{a_1 \dots a_6, b}, \\ Z_{\alpha\beta\gamma}^{a_1 \dots a_9}, Z_{\alpha}^{a_1 \dots a_9}, \hat{Z}_{\alpha}^{a_1 \dots a_9}, Z_{\alpha}^{a_1 \dots a_8, b}, \hat{Z}_{\alpha}^{a_1 \dots a_8, b}, Z_{\alpha}^{a_1 \dots a_7, b_1 b_2} \end{aligned} \quad (3.2.3)$$

We note that some of the l_1 generators at level five have multiplicity two. Although we have listed the generators up to level five we will only use the generators up to level four, that is, we will work just up to level four in what follows.

Lower case Greek indexes correspond to the fundamental representation of $SL(2, R)$ ($\alpha, \beta, \gamma, \dots = 1, 2$). Tensors that have multiple Greek indexes are assumed to be symmetric in these indices. The general group element $g = g_l g_A$, up to level 4, can be written as

$$\begin{aligned} g_l = \exp(x^a P_a + x_a^{\alpha} Z_{\alpha}^a + x_{a_1 a_2 a_3} Z^{a_1 a_2 a_3} + x_{a_1 \dots a_5}^{\alpha} Z_{\alpha}^{a_1 \dots a_5} + x_{a_1 \dots a_7}^{\alpha\beta} Z_{\alpha\beta}^{a_1 \dots a_7} \\ + x_{a_1 \dots a_7} Z^{a_1 \dots a_7} + x_{a_1 \dots a_6, b} Z^{a_1 \dots a_6, b}), \\ g_A = \exp(h_a{}^b K^a{}_b) \exp(\varphi^{\alpha\beta} R_{\alpha\beta}) \exp(A_{a_1 \dots a_7, b} K^{a_1 \dots a_7, b}) \exp(A_{a_1 \dots a_8}^{\alpha\beta} R_{\alpha\beta}^{a_1 \dots a_8}) \\ \times \exp(A_{a_1 \dots a_6}^{\alpha} R_{\alpha}^{a_1 \dots a_6}) \exp(A_{a_1 \dots a_4} R^{a_1 \dots a_4}) \exp(A_{a_1 a_2}^{\alpha} R_{\alpha}^{a_1 a_2}), \end{aligned} \quad (3.2.4)$$

Where we have introduced the generalised coordinates

$$x^a; x_a^{\alpha}; x_{a_1 a_2 a_3}; x_{a_1 \dots a_5}^{\alpha}; x_{a_1 \dots a_7}^{\alpha\beta}; x_{a_1 \dots a_7}; x_{a_1 \dots a_6, b}. \quad (3.2.5)$$

The tangent space group is $I_c(E_{11})$ which at level zero is $SO(1, 9) \times SO(2)$. It is very straightforward to compute at higher levels.

In this section we are going to calculate the generalised vielbein using its definition in equation (1.10) rather than the matrix method of section two. In this approach the generalised vielbein is computed by conjugating the l_1 generators with the E_{11} group element. We recall that

$$E_{\Pi}{}^A l_A = g_A^{-1} l_{\Pi} g_A. \quad (3.2.5)$$

Using the algebra from Appendix A.2 we can perform this conjugation for the $D = 10$ case. Conjugation with level 0 group element gives

$$\begin{aligned} & \exp(-\varphi^{\alpha\beta} R_{\alpha\beta}) \exp(-h_a{}^b K_a{}^b) \left\{ P_\mu, Z_{\dot{\alpha}}^\mu, Z^{\mu_1\mu_2\mu_3}, Z_{\dot{\alpha}}^{\mu_1\cdots\mu_5}, Z_{\dot{\alpha}_1\dot{\alpha}_2}^{\mu_1\cdots\mu_7}, Z^{\mu_1\cdots\mu_7}, Z^{\mu_1\cdots\mu_6,\nu} \right\} \\ & \quad \times \exp(h_a{}^b K_a{}^b) \exp(\varphi^{\alpha\beta} R_{\alpha\beta}) = \\ & = (\det e)^{-\frac{1}{2}} \left\{ e_\mu{}^a P_a, (e^{-1})_a{}^\mu g_{\dot{\alpha}}{}^\beta Z_\beta^a, (e^{-1})_{a_1 a_2 a_3}^{\mu_1 \mu_2 \mu_3} Z^{a_1 a_2 a_3}, (e^{-1})_{a_1 \dots a_5}^{\mu_1 \dots \mu_5} g_{\dot{\alpha}}{}^\beta Z_\beta^{a_1 \dots a_5}, \right. \\ & \quad \left. (e^{-1})_{a_1 \dots a_7}^{\mu_1 \dots \mu_7} g_{\dot{\alpha}_1 \dot{\alpha}_2}^{\beta_1 \beta_2} Z_{\beta_1 \beta_2}^{a_1 \dots a_7}, (e^{-1})_{a_1 \dots a_7}^{\mu_1 \dots \mu_7} Z^{a_1 \dots a_7}, (e^{-1})_{a_1 \dots a_6, b}^{\mu_1 \dots \mu_6, \nu} Z^{a_1 \dots a_6, b} \right\}, \quad (3.2.7) \end{aligned}$$

where $e_\mu{}^b = (e^h)_\mu{}^b$, $g_{\dot{\alpha}}{}^\beta = (e^{\varepsilon_{\bullet\gamma}\varphi^{\gamma\bullet}})_{\dot{\alpha}}{}^\beta$ and

$$\begin{aligned} (e^{-1})_{a_1 \dots a_n}^{\mu_1 \dots \mu_n} &= (e^{-1})_{[a_1}^{\mu_1} \dots (e^{-1})_{a_n]}^{\mu_n}, \quad g_{\alpha_1 \dots \alpha_n}^{\beta_1 \dots \beta_n} = g_{[\alpha_1}^{\beta_1} \dots g_{\alpha_n]}^{\beta_n}, \\ (e^{-1})_{a_1 \dots a_6, b}^{\mu_1 \dots \mu_6, \nu} &= (e^{-1})_{[a_1}^{\mu_1} \dots (e^{-1})_{a_6]}^{\mu_6} (e^{-1})_b{}^\nu - (e^{-1})_{[a_1}^{\mu_1} \dots (e^{-1})_{a_6}^{\mu_6} (e^{-1})_{b]}{}^\nu. \end{aligned} \quad (3.2.8)$$

In the above equation and what follows we denote world, rather than tangent, $SL(2)$ indices with a dot, that is $\dot{\alpha}, \dots$. Conjugating with positive level generators can be obtained by Taylor-expanding the exponents and truncating the series by level 4. For level one E_{11} generator we have

$$\begin{aligned} & \exp(-A_{b_1 b_2}^\alpha R_{\alpha}^{b_1 b_2}) \left\{ P_a, Z_\alpha^{a_1}, Z^{a_1 a_2 a_3}, Z_\alpha^{a_1 \dots a_5} \right\} \exp(A_{b_1 b_2}^\alpha R_{\alpha}^{b_1 b_2}) = \\ & = P_a - A_{ab}^\alpha Z_\alpha^b + \frac{1}{2} \varepsilon_{\alpha\beta} A_{aa_1}^\alpha A_{a_2 a_3}^\beta Z^{a_1 a_2 a_3} - \frac{1}{6} \varepsilon_{\alpha\beta} A_{aa_1}^\alpha A_{a_2 a_3}^\beta A_{a_4 a_5}^\gamma Z_\gamma^{a_1 \dots a_5} + \\ & + \frac{1}{24} \varepsilon_{\alpha\beta} A_{aa_1}^\alpha A_{a_2 a_3}^\beta A_{a_4 a_5}^{\alpha_1} A_{a_6 a_7}^{\alpha_2} Z_{\alpha_1 \alpha_2}^{a_1 \dots a_7} - \frac{1}{60} \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} A_{aa_1}^\alpha A_{a_2 a_3}^\beta A_{a_4 a_5}^\sigma A_{a_6 b}^\lambda Z^{a_1 \dots a_6, b}, \\ & Z_\alpha^{a_1} - \varepsilon_{\alpha\beta} A_{a_2 a_3}^\beta Z^{a_1 a_2 a_3} + \frac{1}{2} \varepsilon_{\alpha\beta} A_{a_2 a_3}^\beta A_{a_4 a_5}^\gamma Z_\gamma^{a_1 \dots a_5} - \frac{1}{6} \varepsilon_{\alpha\beta} A_{a_2 a_3}^\beta A_{a_4 a_5}^{\alpha_1} A_{a_6 a_7}^{\alpha_2} Z_{\alpha_1 \alpha_2}^{a_1 \dots a_7} + \\ & + \frac{1}{15} \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} A_{a_2 a_3}^\beta A_{a_4 a_5}^\sigma A_{a_6 b}^\lambda Z^{a_1 \dots a_6, b}, Z^{a_1 a_2 a_3} - A_{a_4 a_5}^\alpha Z_\alpha^{a_1 \dots a_5} + \frac{1}{2} A_{a_4 a_5}^{\alpha_1} A_{a_6 a_7}^{\alpha_2} Z_{\alpha_1 \alpha_2}^{a_1 \dots a_7} - \\ & - \frac{1}{5} \varepsilon_{\alpha\beta} A_{a_4 a_5}^\alpha A_{a_6 b}^\beta Z^{a_1 \dots a_6, b}, Z_\alpha^{a_1 \dots a_5} - A_{a_6 a_7}^\beta Z_{\alpha\beta}^{a_1 \dots a_7} - \varepsilon_{\alpha\beta} A_{a_6 a_7}^\beta Z^{a_1 \dots a_7} + \frac{2}{5} \varepsilon_{\alpha\beta} A_{a_6 b}^\beta Z^{a_1 \dots a_6, b}. \end{aligned} \quad (3.2.9)$$

For level 2 E_{11} generator:

$$\begin{aligned} & \exp(-A_{b_1 \dots b_4} R^{b_1 \dots b_4}) \left\{ P_a, Z_\alpha^{a_1}, Z^{a_1 a_2 a_3} \right\} \exp(A_{b_1 \dots b_4} R^{b_1 \dots b_4}) = \\ & = P_a - 2 A_{aa_1 a_2 a_3} Z^{a_1 a_2 a_3} + 2 A_{aa_1 a_2 a_3} A_{a_4 \dots a_7} Z^{a_1 \dots a_7} - \frac{4}{5} A_{aa_1 a_2 a_3} A_{a_4 a_5 a_6 b} Z^{a_1 \dots a_6, b}, \end{aligned}$$

$$Z_\alpha^{a_1} + A_{a_2 \dots a_5} Z_\alpha^{a_1 \dots a_5}, Z^{a_1 a_2 a_3} - 2 A_{a_4 \dots a_7} Z^{a_1 \dots a_7} + \frac{4}{5} A_{a_4 a_5 a_6 b} Z^{a_1 \dots a_6, b}. \quad (3.2.10)$$

For level 3 E_{11} generator:

$$\begin{aligned} & \exp \left(-A_{b_1 \dots b_6}^\beta R_\beta^{b_1 \dots b_6} \right) \left\{ P_a, Z_\alpha^{a_1} \right\} \exp \left(A_{b_1 \dots b_6}^\beta R_\beta^{b_1 \dots b_6} \right) = \\ & = P_a - \frac{3}{4} A_{a a_1 \dots a_5}^\alpha Z_\alpha^{a_1 \dots a_5}, Z_\alpha^{a_1} + \frac{1}{4} A_{a_2 \dots a_7}^\beta Z_{\alpha \beta}^{a_1 \dots a_7} + \\ & + \frac{3}{4} \varepsilon_{\alpha \beta} A_{a_2 \dots a_7}^\beta Z^{a_1 \dots a_7} + \frac{1}{20} \varepsilon_{\alpha \beta} A_{a_2 \dots a_7}^\beta Z^{a_2 \dots a_7, a_1}. \end{aligned} \quad (3.2.11)$$

and, finally, for level E_{11} 4 generators:

$$\exp \left(-A_{b_1 \dots b_8}^{\beta_1 \beta_2} R_{\beta_1 \beta_2}^{b_1 \dots b_8} \right) P_a \exp \left(A_{b_1 \dots b_8}^{\beta_1 \beta_2} R_{\beta_1 \beta_2}^{b_1 \dots b_8} \right) = P_a + A_{a a_1 \dots a_7}^{\alpha_1 \alpha_2} Z_{\alpha_1 \alpha_2}^{a_1 \dots a_7}, \quad (3.2.12)$$

$$\begin{aligned} & \exp \left(-A_{b_1 \dots b_7, b} R^{b_1 \dots b_7, b} \right) P_a \exp \left(A_{b_1 \dots b_7, b} R^{b_1 \dots b_7, b} \right) \\ & = P_a + 3 A_{a_1 \dots a_7, a} Z^{a_1 \dots a_7} - \frac{21}{20} A_{a a_1 \dots a_6, b} Z^{a_1 \dots a_6, b}. \end{aligned}$$

Using all these results we find, from equation (3.2.5), that the generalised vielbein is given by

$$E_\Pi^A = (\det e)^{-\frac{1}{2}}$$

$$\begin{pmatrix} e_\mu^a & \alpha_\mu|_a^\beta & \alpha_\mu|_{a_1 a_2 a_3} & \alpha_\mu|_{a_1 \dots a_5}^\beta & \alpha_\mu|_{a_1 \dots a_7}^{\beta_1 \beta_2} & \alpha_\mu|_{a_1 \dots a_7} & \alpha_\mu|_{a_1 \dots a_6, c} \\ 0 & (e^{-1})_a^\mu g_{\dot{\alpha}}^\beta & \beta_{\dot{\alpha}}^\mu|_{a_1 a_2 a_3} & \beta_{\dot{\alpha}}^\mu|_{a_1 \dots a_5}^\beta & \beta_{\dot{\alpha}}^\mu|_{a_1 \dots a_7}^{\beta_1 \beta_2} & \beta_{\dot{\alpha}}^\mu|_{a_1 \dots a_7} & \beta_{\dot{\alpha}}^\mu|_{a_1 \dots a_6, c} \\ 0 & 0 & (e^{-1})_{a_1 a_2 a_3}^{\mu_1 \mu_2 \mu_3} & \gamma^{\mu_1 \mu_2 \mu_3} \beta_{|a_1 \dots a_5}^\beta & \gamma^{\mu_1 \mu_2 \mu_3} \beta_{|a_1 \dots a_7}^{\beta_1 \beta_2} & \gamma^{\mu_1 \mu_2 \mu_3} |_{a_1 \dots a_7} & \gamma^{\mu_1 \mu_2 \mu_3} |_{a_1 \dots a_6, c} \\ 0 & 0 & 0 & (e^{-1})_{a_1 \dots a_5}^{\mu_1 \dots \mu_5} g_{\dot{\alpha}}^\beta & \chi_{\dot{\alpha}}^{\mu_1 \dots \mu_5} \beta_{|a_1 \dots a_7}^{\beta_1 \beta_2} & \chi_{\dot{\alpha}}^{b_1 \dots b_5} |_{a_1 \dots a_7} & \chi_{\dot{\alpha}}^{\mu_1 \dots \mu_5} |_{a_1 \dots a_6, c} \\ 0 & 0 & 0 & 0 & (e^{-1})_{a_1 \dots a_7}^{\mu_1 \dots \mu_7} g_{\dot{\alpha} \dot{\alpha}_2}^{\beta_1 \beta_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (e^{-1})_{a_1 \dots a_7}^{\mu_1 \dots \mu_7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (e^{-1})_{a_1 \dots a_6, c}^{\mu_1 \dots \mu_6, \nu} \end{pmatrix} \quad (3.2.13)$$

In the above world indices, that is, μ, \dots or $\dot{\alpha}, \dots$ arise, as vielbeins acting on objects with tangent indices, for example

$$\alpha_\mu|_a^\beta = e_\mu^b \alpha_b|_a^\beta, \quad \beta_{\dot{\alpha}}^\mu|_{a_1 a_2 a_3} = (e^{-1})_b^\mu g_{\dot{\alpha}}^\gamma \beta_\gamma^{b_1 b_2 b_3},$$

$$\gamma^{\mu_1 \mu_2 \mu_3} \beta_{|a_1 \dots a_5}^\beta = (e^{-1})_{b_1 b_2 b_3}^{\mu_1 \mu_2 \mu_3} \gamma^{b_1 b_2 b_3} \beta_{|a_1 \dots a_5}^\beta,$$

$$\chi_{\dot{\alpha}}^{\mu_1 \dots \mu_5} \beta_{|a_1 \dots a_7}^{\beta_1 \beta_2} = (e^{-1})_{b_1 \dots b_5}^{\mu_1 \dots \mu_5} g_{\dot{\alpha}}^\gamma \chi_\gamma^{b_1 \dots b_5} \beta_{|a_1 \dots a_7}^{\beta_1 \beta_2}, \dots$$

$$\chi_{\dot{\alpha}}^{\mu_1 \dots \mu_5} |_{a_1 \dots a_7} = (e^{-1})_{b_1 \dots b_5}^{\mu_1 \dots \mu_5} g_{\dot{\alpha}}^\gamma \chi_\gamma^{b_1 \dots b_5} |_{a_1 \dots a_7},$$

$$\chi_{\dot{\alpha}}^{\mu_1 \dots \mu_5}|_{a_1 \dots a_6, b} = (e^{-1})_{b_1 \dots b_5}^{\mu_1 \dots \mu_5} g_{\dot{\alpha}}^{\gamma} \chi_{\gamma}^{b_1 \dots b_5}|_{a_1 \dots a_6, b}, \dots$$

The symbols in the first line of the above matrix are given by

$$\begin{aligned} \alpha_a|_b^{\alpha} &= -A_{ab}^{\alpha}, \quad \alpha_{a|a_1 a_2 a_3} = -2 A_{aa_1 a_2 a_3} + \frac{1}{2} \varepsilon_{\alpha\beta} A_{a[a_1}^{\alpha} A_{a_2 a_3]}^{\beta}, \\ \alpha_a|_{a_1 \dots a_5}^{\alpha} &= -\frac{3}{4} A_{aa_1 \dots a_5}^{\alpha} + 2 A_{a[a_1 a_2 a_3} A_{a_4 a_5]}^{\alpha} - \frac{1}{6} \varepsilon_{\beta\gamma} A_{a[a_1}^{\beta} A_{a_2 a_3}^{\gamma} A_{a_4 a_5]}^{\alpha}, \\ \alpha_a|_{a_1 \dots a_7}^{\alpha_1 \alpha_2} &= A_{aa_1 \dots a_7}^{\alpha_1 \alpha_2} + \frac{3}{4} A_{a[a_1 \dots a_5}^{(\alpha_1} A_{a_6 a_7]}^{\alpha_2)} - A_{a[a_1 a_2 a_3} A_{a_4 a_5}^{\alpha_1} A_{a_6 a_7]}^{\alpha_2} \\ &\quad + \frac{1}{24} \varepsilon_{\beta\gamma} A_{a[a_1}^{\beta} A_{a_2 a_3}^{\gamma} A_{a_4 a_5}^{\alpha_1} A_{a_6 a_7]}^{\alpha_2}, \\ \alpha_{a|a_1 \dots a_7} &= 3 A_{a_1 \dots a_7, a} + \frac{3}{4} \varepsilon_{\alpha\beta} A_{a[a_1 \dots a_5}^{\alpha} A_{a_6 a_7]}^{\beta} + 2 A_{a[a_1 a_2 a_3} A_{a_4 \dots a_7]}, \\ \alpha_{a|a_1 \dots a_6, b} &= -\frac{21}{20} A_{aa_1 \dots a_6, b} - \frac{3}{10} \varepsilon_{\alpha\beta} A_{a[a_1 \dots a_5}^{\alpha} A_{a_6]b}^{\beta} + \frac{3}{10} \varepsilon_{\alpha\beta} A_{a[a_1 \dots a_5}^{\alpha} A_{a_6 b]}^{\beta} \\ &\quad - \frac{4}{5} A_{a[a_1 a_2 a_3} A_{a_4 a_5 a_6]b} + \frac{4}{5} A_{a[a_1 a_2 a_3} A_{a_4 a_5 a_6 b]} + \frac{2}{5} \varepsilon_{\alpha\beta} A_{a[a_1 a_2 a_3} A_{a_4 a_5}^{\alpha} A_{a_6]b}^{\beta} \\ &\quad - \frac{1}{60} \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} A_{a[a_1}^{\alpha} A_{a_2 a_3}^{\beta} A_{a_4 a_5}^{\sigma} A_{a_6]b}^{\lambda}, \end{aligned} \tag{3.2.14}$$

in the second line are

$$\begin{aligned} \beta_{\alpha}^a|_{a_1 a_2 a_3} &= -\varepsilon_{\alpha\beta} \delta_{[a_1}^a A_{a_2 a_3]}^{\beta}, \quad \beta_{\alpha}^a|_{a_1 \dots a_5}^{\beta} = \delta_{\alpha}^{\beta} \delta_{[a_1}^a A_{a_2 \dots a_5]} + \frac{1}{2} \varepsilon_{\alpha\gamma} \delta_{[a_1}^a A_{a_2 a_3}^{\gamma} A_{a_4 a_5]}^{\beta}, \\ \beta_{\alpha}^a|_{a_1 \dots a_7}^{\beta_1 \beta_2} &= \frac{1}{4} \delta_{[a_1}^a \delta_{\alpha}^{(\beta_1} A_{a_2 \dots a_7]}^{\beta_2)} - \delta_{[a_1}^a \delta_{\alpha}^{(\beta_1} A_{a_2 \dots a_5} A_{a_6 a_7]}^{\beta_2)} - \frac{1}{6} \varepsilon_{\alpha\gamma} \delta_{[a_1}^a A_{a_2 a_3}^{\gamma} A_{a_4 a_5}^{\beta_1} A_{a_6 a_7]}^{\beta_2}, \\ \beta_{\alpha}^a|_{a_1 \dots a_7} &= \frac{3}{4} \varepsilon_{\alpha\beta} \delta_{[a_1}^a A_{a_2 \dots a_7]}^{\beta} - \varepsilon_{\alpha\beta} \delta_{[a_1}^a A_{a_2 \dots a_5} A_{a_6 a_7]}^{\beta}, \quad \beta_{\alpha}^a|_{a_1 \dots a_6, b} = \frac{1}{20} \varepsilon_{\alpha\beta} \delta_b^a A_{a_1 \dots a_6}^{\beta} - \\ &\quad - \frac{1}{20} \varepsilon_{\alpha\beta} \delta_{[b}^a A_{a_1 \dots a_6]}^{\beta} + \frac{2}{5} \varepsilon_{\alpha\beta} \delta_{[a_1}^a A_{a_2 \dots a_5} A_{a_6]b}^{\beta} \\ &\quad - \frac{2}{5} \varepsilon_{\alpha\beta} \delta_{[a_1}^a A_{a_2 \dots a_5} A_{a_6 b]}^{\beta} + \frac{1}{15} \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} \delta_{[a_1}^a A_{a_2 a_3}^{\beta} A_{a_4 a_5}^{\sigma} A_{a_6]b}^{\lambda}, \end{aligned} \tag{3.2.15}$$

in the third line are

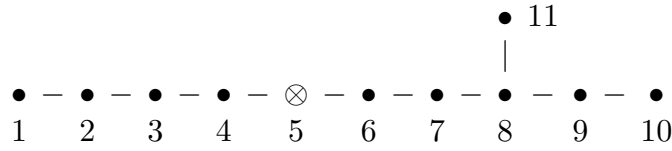
$$\begin{aligned} \gamma^{b_1 b_2 b_3}|_{a_1 \dots a_5}^{\beta} &= -\delta_{[a_1 a_2 a_3}^{\beta} A_{a_4 a_5]}^{\beta}, \quad \gamma^{b_1 b_2 b_3}|_{a_1 \dots a_7}^{\beta_1 \beta_2} = \frac{1}{2} \delta_{[a_1 a_2 a_3}^{\beta_1} A_{a_4 a_5}^{\beta_2} A_{a_6 a_7]}^{\beta}, \\ \gamma^{b_1 b_2 b_3}|_{a_1 \dots a_7} &= -2 \delta_{[a_1 a_2 a_3}^{\beta} A_{a_4 \dots a_7]}^{\beta}, \quad \gamma^{b_1 b_2 b_3}|_{a_1 \dots a_6, b} = \frac{4}{5} \delta_{[a_1 a_2 a_3}^{\beta} A_{a_4 a_5 a_6]b}^{\beta} - \\ &\quad - \frac{4}{5} \delta_{[a_1 a_2 a_3}^{\beta} A_{a_4 a_5 a_6 b]}^{\beta} - \frac{1}{5} \varepsilon_{\alpha\beta} \delta_{[a_1 a_2 a_3}^{\beta} A_{a_4 a_5}^{\alpha} A_{a_6]b}^{\beta}. \end{aligned} \tag{3.2.16}$$

and finally in the fourth line are

$$\begin{aligned}\chi_{\alpha}^{b_1 \dots b_5} |_{a_1 \dots a_7}^{\alpha_1 \alpha_2} &= -\delta_{\alpha}^{(\alpha_1} \delta_{[a_1 \dots a_5}^{b_1 \dots b_5} A_{a_6 a_7]}^{\alpha_2)}, \quad \chi_{\alpha}^{b_1 \dots b_5} |_{a_1 \dots a_7} = -\varepsilon_{\alpha \beta} \delta_{[a_1 \dots a_5}^{b_1 \dots b_5} A_{a_6 a_7]}^{\beta}, \\ \chi_{\alpha}^{b_1 \dots b_5} |_{a_1 \dots a_6, b} &= -\varepsilon_{\alpha \beta} \delta_{[a_1 \dots a_5}^{b_1 \dots b_5} A_{a_6] b}^{\beta} + \varepsilon_{\alpha \beta} \delta_{[a_1 \dots a_5}^{b_1 \dots b_5} A_{a_6 b]}^{\beta}.\end{aligned}\quad (3.2.17)$$

3.3 $D = 5$

The five dimensional theory is obtained by deleting node 5 from the E_{11} Dynkin diagram, given below, to find the algebra $GL(5) \times E_6$ and decomposing the $E_{11} \otimes_s l_1$ algebra into representations of this algebra [15].



In this decomposition the positive, including zero, level generators of the E_{11} algebra are

$$\begin{aligned}K^a_b, R^{\alpha}, R^{aM}, R^{a_1 a_2}_N, R^{a_1 a_2 a_3, \alpha}, \\ R^{a_1 a_2, b}, R^{a_1 \dots a_4}_{N_1 N_2}, R^{a_1, b_1 b_2 b_3}_N, \dots\end{aligned}\quad (3.3.1)$$

where $R^{[a_1 a_2, b]} = 0$ and $R^{[a_1, b_1 b_2 b_3] N} = 0$, while those with negative level are given by

$$R_{aM}, R_{a_1 a_2}^N, R_{a_1 a_2 a_3}^{\alpha}, R_{a_1 a_2, b}, R_{a_1 \dots a_4}^{N_1 N_2}, R_{a_1, b_1 b_2 b_3 N}, \dots\quad (3.3.2)$$

The l_1 representation decomposes to give the generators [15]

$$P_a, Z^N, Z^a_N, Z^{a_1 a_2, \alpha}, Z^{a_1 a_2}, Z^{a_1 a_2, b N}, Z^{a_1 a_2 a_3}_{N_1 N_2}, \dots\quad (3.3.3)$$

The fifth generator does not obey $Z^{[a_1 a_2, b] N} = 0$ and the third generator $Z^{a_1 a_2}$ has no symmetries on its two indices. For these objects the lower case Latin indexes correspond to 5-dimensional representation of $GL(5)$ ($a, b, c, \dots = 1, \dots, 5$). Greek indexes correspond to 78-dimensional adjoint representation of E_6 ($\alpha, \beta, \gamma, \dots = 1, \dots, 78$). Upper and lower case Latin indexes correspond to $\bar{27}$ -dimensional and 27-dimensional representations respectively of E_6 ($N, M, P, \dots = 1, \dots, 27$). The 351-dimensional representation can be written as two antisymmetrised indices ie X_{NM} .

An arbitrary group element can be parametrised in the following way:

$$\begin{aligned}g_l &= \exp(x^a P_a + x_N Z^N + x_a^N Z^a_N + x_{a_1 a_2, \alpha} Z^{a_1 a_2, \alpha} + x_{ab} Z^{ab}), \\ g_A &= \exp(h_a^b K^a_b) \exp(\varphi_{\alpha} R^{\alpha}) \exp(A_{a_1 a_2 a_3, \alpha} R^{a_1 a_2 a_3, \alpha}) \times \\ &\times \exp(A_{a_1 a_2, b} R^{a_1 a_2, b}) \exp(A_{a_1 a_2}^N R^{a_1 a_2}_N) \exp(A_{aN} R^{aN}).\end{aligned}\quad (3.3.4)$$

We find that the five dimensional theory has a generalised space-time that has the coordinates

$$x^a, x_N, x_a^N, x_{a_1 a_2, \alpha}, x_{ab}, \dots \quad (3.3.5)$$

and the fields

$$h_a^b, \varphi_\alpha, A_{aM}, A_{a_1 a_2}^N, A_{a_1 a_2 a_3, \alpha}, A_{a_1 a_2, b}, \dots \quad (3.3.6)$$

which depend on the generalised space-time. The tangent space structure is obvious from the presence of the coordinates and the tangent space group is $I_c(E_{11})$ which are lowest level is $SO(1, 4) \otimes Usp(8)$. The generalised vierbein is defined in equation (1.10) and it is straight forward, using the commutators in appendix A.3, to find the generalised vielbein. However, one can also the matrix expression of equation (2.1), or more appropriately equation (2.10), which in the five dimensional case takes the form

$$E_\Pi^A = e^{A_0} e^{\tilde{A}_0} e^{A_3} e^{\tilde{A}_3} e^{A_2} e^{A_1}, \quad (3.3.7)$$

where

$$\begin{aligned} \mathcal{A}_0 &\equiv h_a^b D_a^b, \quad \tilde{\mathcal{A}}_0 \equiv \varphi_\alpha D^\alpha, \quad \mathcal{A}_1 \equiv A_{aN} D^{aN}, \quad \mathcal{A}_2 \equiv A_{a_1 a_2}^N D^{a_1 a_2}_N, \\ \mathcal{A}_3 &\equiv A_{a_1 a_2 a_3, \alpha} D^{a_1 a_2 a_3, \alpha}, \quad \tilde{\mathcal{A}}_3 \equiv A_{a_1 a_2, b} D^{a_1 a_2, b}, \end{aligned} \quad (3.3.8)$$

We will compute the generalised vielbein up to level three. We begin by considering the level zero part and noting that

$$\begin{aligned} dx \cdot (\mathcal{A}_0) \cdot l = \\ - [h_a^b K^a_b, dx^a P_a + dx_N Z^N + dx_a^N Z^a_N + dx_{a_1 a_2, \alpha} Z^{a_1 a_2, \alpha} + dx_{ab} Z^{ab}], \end{aligned} \quad (3.3.9)$$

and

$$\begin{aligned} dx \cdot (\tilde{\mathcal{A}}_0) \cdot l = \\ - [\varphi_\alpha R^\alpha, dx^a P_a + dx_N Z^N + dx_a^N Z^a_N + dx_{a_1 a_2, \alpha} Z^{a_1 a_2, \alpha} + dx_{ab} Z^{ab}], \end{aligned} \quad (3.3.10)$$

from which we conclude that

$$\mathcal{A}_0 = \begin{pmatrix} h_a^b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -h_b^a \delta_N^M & 0 & 0 \\ 0 & 0 & 0 & -2 h_{[b_1}^{[a_1} \delta_{b_2]}^{a_2]} \delta_\beta^\alpha & 0 \\ 0 & 0 & 0 & 0 & -h_c^a \delta_d^b - \delta_c^a h_d^b \end{pmatrix} - \frac{1}{2} h_e^e I, \quad (3.3.11)$$

and

$$\tilde{\mathcal{A}}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\varphi_\alpha (D^\alpha)_M^N & 0 & 0 & 0 \\ 0 & 0 & \delta_b^a \varphi_\alpha (D^\alpha)_N^M & 0 & 0 \\ 0 & 0 & 0 & \delta_{b_1 b_2}^{a_1 a_2} \varphi_\gamma f^{\gamma \alpha}_\beta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.3.12)$$

It then follows that

$$e^{A_0} e^{\tilde{A}_0} = (\det e)^{-\frac{1}{2}} \times$$

$$\begin{pmatrix} e_\mu^a & 0 & 0 & 0 & 0 \\ 0 & (d^{-1})_M^{\dot{N}} & 0 & 0 & 0 \\ 0 & 0 & d_{\dot{N}}^M (e^{-1})_a^\mu & 0 & 0 \\ 0 & 0 & 0 & (e^{-1})_{a_1 a_2}^{\mu_1 \mu_2} (f^{-1})_\beta^{\dot{\alpha}} & 0 \\ 0 & 0 & 0 & 0 & (e^{-1})_a^\mu (e^{-1})_b^\nu \end{pmatrix}, \quad (3.3.13)$$

where

$$e_\mu^a = (e^h)_\mu^a, \quad d_{\dot{N}}^M = (e^{\varphi^\alpha D_\alpha})_{\dot{N}}^M, \quad f_{\dot{\alpha}}^\beta = (e^{\varphi_\gamma f^{\gamma \bullet}})_{\dot{\alpha}}^\beta, \quad (3.3.14)$$

and

$$(e^{-1})_{a_1 \dots a_n}^{\mu_1 \dots \mu_n} = (e^{-1})_{[a_1}^{\mu_1} \dots (e^{-1})_{a_n]}^{\mu_n}, \quad d_{\dot{N}_1 \dots \dot{N}_n}^{M_1 \dots M_n} = d_{[\dot{N}_1}^{M_1} \dots d_{\dot{N}_n]}^{M_n}. \quad (3.3.15)$$

A dot over an index means that it is a world rather than a tangent index.

We now compute \mathcal{A}_1 in a similar way by considering

$$dx \cdot (\mathcal{A}_1) \cdot l = - [A_{aN} R^{aN}, dx^a P_a + dx_N Z^N + dx_a^N Z^a_N + dx_{a_1 a_2, \alpha} Z^{a_1 a_2, \alpha} + dx_{ab} Z^{ab}], \quad (3.3.16)$$

from which we conclude, using the commutators of appendix A.3, that

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -A_{aM} & 0 & 0 & 0 \\ 0 & 0 & d^{NMP} A_{bP} & 0 & 0 \\ 0 & 0 & 0 & -(D_\beta)_N^M \delta_{[b_1}^a A_{b_2]M} A_{cN} \delta_d^a & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.3.17)$$

Proceeding in a similar way we find that

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 0 & -2 A_{ab}^M & 0 & 0 \\ 0 & 0 & 0 & (D_\beta)_P^N A_{b_1 b_2}^P - 2 A_{cd}^N & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.3.18)$$

$$\mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 & -3 A_{ab_1 b_2, \beta} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.3.19)$$

and

$$\tilde{\mathcal{A}}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & -4 A_{d(a, c)} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.3.20)$$

It is now just a matter of matrix multiplication, albeit with unusual index sets, to find the generalised vielbein using equation (3.3.7), the result is

$$E_{\Pi}^A = (\det e)^{-\frac{1}{2}} \begin{pmatrix} e_{\mu}^a & e_{\mu}^b \alpha_{b|M} & e_{\mu}^b \alpha_{b|a}^M & e_{\mu}^b \alpha_{b|a_1 a_2, \alpha} & e_{\mu}^b \alpha_{b|cd} \\ 0 & (d^{-1})_M^{\dot{N}} & (d^{-1})_P^{\dot{N}} \beta_{\dot{a}}^P{}^M & (d^{-1})_P^{\dot{N}} \beta_{\dot{a} a_2, \alpha}^P & (d^{-1})_P^{\dot{N}} \beta_{cd}^P \\ 0 & 0 & d_{\dot{N}}^M (e^{-1})_a^{\mu} & d_{\dot{N}}^P (e^{-1})_b^{\mu} \gamma_{P|a_1 a_2, \alpha}^b & d_{\dot{N}}^P (e^{-1})_b^{\mu} \gamma_{P|cd}^b \\ 0 & 0 & 0 & (e^{-1})_{a_1 a_2}^{\mu_1 \mu_2} (f^{-1})_{\beta}^{\alpha} & 0 \\ 0 & 0 & 0 & 0 & (e^{-1})_c^{\mu} (e^{-1})_d^{\nu} \end{pmatrix}, \quad (3.3.21)$$

where in the first line

$$\begin{aligned} \alpha_{a|N} &= -A_{aN}, \quad \alpha_{a|b}^N = -2A_{ab}^N - \frac{1}{2} d^{NMP} A_{aM} A_{bP}, \\ \alpha_{a|a_1 a_2, \alpha} &= -3A_{aa_1 a_2, \alpha} + 2A_{a[a_1}^N A_{a_2]M} (D_{\alpha})_N^M + \frac{1}{6} A_{aN} A_{[a_1 M} A_{a_2]P} d^{NMS} (D_{\alpha})_S^P, \\ \alpha_{a|cd} &= -4A_{d(a, c)} - 2A_{ad}^N A_{cN} - \frac{1}{6} A_{aN} A_{bM} A_{cP} d^{NMP}, \end{aligned} \quad (3.3.22)$$

in the second line

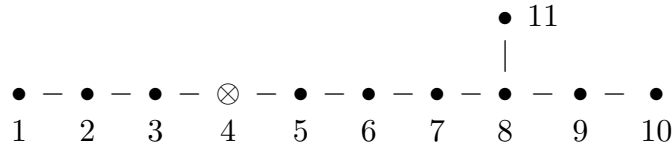
$$\begin{aligned} \beta_{\dot{a}}^N{}^M &= A_{aP} d^{NMP}, \quad \beta_{\dot{a} a_1 a_2, \alpha}^N = A_{a_1 a_2}^M (D_{\alpha})_M^N - \frac{1}{2} A_{[a_1 M} A_{a_2]R} d^{NMP} (D_{\alpha})_P^R, \\ \beta_{ab}^N &= -2A_{ab}^N + \frac{1}{2} A_{aM} A_{bP} d^{NMP}, \end{aligned} \quad (3.3.23)$$

and in the third line

$$\gamma^a{}_{N|a_1 a_2, \alpha} = -\delta_{[a_1}^a A_{a_2]M} (D_{\alpha})_N^M, \quad \gamma^a{}_{N|cd} = \delta_d^a A_{cN}. \quad (3.3.24)$$

3.4 $D = 4$

The four dimensional theory is obtained by deleting node 4 from the Dynkin diagram and so decomposing the E_{11} algebra into representations of $GL(4) \times E_7$ [19]. However, it is easier to work with $SL(8)$ subalgebra of E_7 , instead of E_7 itself; the E_7 representations can be reconstructed if needed. In this case all the generators belong to different representations of $GL(4) \times SL(8)$.



In this section we are going to calculate the Cartan form up to level 2. The positive (and zero) level generators of E_{11} are

$$K^a{}_b, R^I{}_J, R^{I_1 \dots I_4}; R^{a I_1 I_2}, R^a{}_{I_1 I_2}; \hat{K}^{(ab)}, R^{a_1 a_2 I}{}_J, R^{a_1 a_2 I_1 \dots I_4}. \quad (3.4.1)$$

The negative level generators are

$$R_{a I_1 I_2}, R_a{}^{I_1 I_2}; \hat{K}_{(ab)}, R_{a_1 a_2}{}^I{}_J, R_{a_1 a_2 I_1 \dots I_4}. \quad (3.4.2)$$

The l_1 representation generators are

$$P_a; Z^{I_1 I_2}, Z_{I_1 I_2}; Z^a, Z^a{}_I, Z^{a I_1 \dots I_4}. \quad (3.4.3)$$

The parametrisation of an arbitrary level 2 group element is of the form

$$\begin{aligned} g_l &= \exp \left(x^a P_a + x_{I_1 I_2} Z^{I_1 I_2} + x^{I_1 I_2} Z_{I_1 I_2} + \hat{x}_a Z^a + x_a{}^J{}_I Z^{a I}{}_J + x_{a I_1 \dots I_4} Z^{a I_1 \dots I_4} \right), \\ g_A &= \exp \left(h_a{}^b K^a{}_b \right) \exp \left(\varphi^I{}_J R^I{}_J \right) \exp \left(\varphi_{I_1 \dots I_4} R^{I_1 \dots I_4} \right) \exp \left(\hat{h}_{(ab)} \hat{K}^{(ab)} \right) \\ &\times \exp \left(A_{a_1 a_2}{}^J{}_I R^{a_1 a_2 I}{}_J \right) \exp \left(A_{a_1 a_2 I_1 \dots I_4} R^{a_1 a_2 I_1 \dots I_4} \right) \exp \left(A_{a I_1 I_2} R^{a I_1 I_2} + A_a{}^{I_1 I_2} R^a{}_{I_1 I_2} \right), \end{aligned} \quad (3.4.4)$$

where we have introduced the generalised coordinates

$$x^a; x_{I_1 I_2}, x^{I_1 I_2}; \hat{x}_a, x_a{}^I{}_J, x_{a I_1 \dots I_4}, \quad (3.4.5)$$

and the fields

$$h_a{}^b, \varphi^I{}_J, \varphi_{I_1 \dots I_4}; A_{a I_1 I_2}, A_a{}^{I_1 I_2}; \hat{h}_{(ab)}, A_{a_1 a_2}{}^I{}_J, A_{a_1 a_2 I_1 \dots I_4}. \quad (3.4.6)$$

To calculate the generalised vielbein we used the definition of equation (1.10), which was the same technique as was used in section (3.2) for the ten dimensional IIB theory. Conjugation of any l_1 generator with group element that contains the $K^a{}_b$ and $R^I{}_J$ generators gives the following:

$$\begin{aligned} &\exp \left(-\varphi^I{}_J R^I{}_J \right) \exp \left(-h_a{}^b K^a{}_b \right) \left\{ P_\mu, Z^{I_1 I_2}, Z_{I_1 I_2}, Z^\mu, Z^{\mu I}{}_J, Z^{\mu I_1 \dots I_4} \right\} \\ &\exp \left(h_a{}^b K^a{}_b \right) \exp \left(\varphi^I{}_J R^I{}_J \right) = \\ &= (\det e)^{-\frac{1}{2}} \left\{ e_\mu{}^a P_a, (f^{-1})^{i_1 i_2}{}_{j_1 j_2} Z^{j_1 j_2}, f^{j_1 j_2}{}_{i_1 i_2} Z_{j_1 j_2}, (e^{-1})_a{}^\mu Z^a, \right. \\ &\quad \left. (e^{-1})_a{}^\mu (f^{-1})^i{}_K f^L{}_J Z^{a K}{}_L, (e^{-1})_a{}^\mu (f^{-1})^{i_1 \dots i_4}{}_{j_1 \dots j_4} Z^{a j_1 \dots j_4} \right\}, \end{aligned} \quad (3.4.7)$$

where $e_\mu{}^b = (e^h)_\mu{}^b$, $f^I{}_J = (e^\varphi)^I{}_J$, and

$$(e^{-1})^{\mu_1 \dots \mu_n}{}_{a_1 \dots a_n} = (e^{-1})^{\mu_1}{}_{[a_1} \dots (e^{-1})^{\mu_n}{}_{a_n]}, \quad (f^{-1})^{i_1 \dots i_n}{}_{j_1 \dots j_n} = f^{i_1}{}_{[j_1} \dots f^{i_n}{}_{j_n]}. \quad (3.4.8)$$

We place a dot on a SL(8) index to denote that it is a world, rather than a tangent, index.

Conjugation with $R^{I_1 \dots I_4}$ generator gives

$$\begin{aligned} \exp(-\varphi_{I_1 \dots I_4} R^{I_1 \dots I_4}) \left\{ P_a, Z^{I_1 I_2}, Z_{I_1 I_2}, Z^a, Z^a{}^I{}_J, Z^{a I_1 \dots I_4} \right\} \exp(\varphi_{I_1 \dots I_4} R^{I_1 \dots I_4}) = \\ = \left\{ P_a, \beta^{I_1 I_2}_{J_1 J_2} Z^{J_1 J_2} + \beta^{I_1 I_2|J_1 J_2} Z_{J_1 J_2}, \beta^{J_1 J_2}_{I_1 I_2} Z_{J_1 J_2} + \beta_{I_1 I_2|J_1 J_2} Z^{J_1 J_2}, Z^a, \right. \\ \left. \gamma^I{}_{J|K}{}^L Z^{aK}{}_L + \gamma^I{}_{J|J_1 \dots J_4} Z^{aJ_1 \dots J_4}, \gamma^{I_1 \dots I_4}_{J_1 \dots J_4} Z^{aJ_1 \dots J_4} + \gamma^{I_1 \dots I_4}{}_K{}^L Z^{aK}{}_L \right\}, \quad (3.4.9) \end{aligned}$$

where the β -matrices that mix level 1 elements are defined as

$$\begin{aligned} \beta^{I_1 I_2}_{J_1 J_2} &= \left(1 + \frac{1}{2} P + \frac{1}{4!} P^2 + \frac{1}{6!} P^3 + \dots \right)_{J_1 J_2}^{I_1 I_2}, \quad P^{I_1 I_2}_{J_1 J_2} = \frac{1}{24} \varepsilon^{I_1 \dots I_8} \varphi_{I_3 \dots I_6} \varphi_{I_7 I_8 J_1 J_2}, \\ \beta^{I_1 I_2|J_1 J_2} &= -\frac{1}{24} \varepsilon^{J_1 \dots J_8} \left(1 + \frac{1}{3!} P + \frac{1}{5!} P^2 + \frac{1}{7!} P^3 + \dots \right)_{J_3 J_4}^{I_1 I_2} \varphi_{J_5 \dots J_8}, \\ \beta_{I_1 I_2|J_1 J_2} &= -\varphi_{J_1 \dots J_4} \left(1 + \frac{1}{3!} P + \frac{1}{5!} P^2 + \frac{1}{7!} P^3 + \dots \right)_{I_1 I_2}^{J_3 J_4}, \quad (3.4.10) \end{aligned}$$

while the γ -matrices, responsible for mixing of level 2 elements, are given by

$$\begin{aligned} \gamma^I{}_{J|K}{}^L &= \left(1 + \frac{1}{2} Q + \frac{1}{4!} Q^2 + \frac{1}{6!} Q^3 + \dots \right)_{J|K}^I{}^L, \\ Q^I{}_{J|K}{}^L &= \left(\frac{1}{72} \delta_J^I \varphi_{I_1 \dots I_4} - \frac{1}{9} \delta_{I_1}^I \varphi_{J I_2 I_3 I_4} \right) \varepsilon^{I_1 \dots I_4 J_1 J_2 J_3 L} \varphi_{J_1 J_2 J_3 K}, \\ \gamma^{I_1 \dots I_4}_{J_1 \dots J_4} &= \left(1 + \frac{1}{2} R + \frac{1}{4!} R^2 + \dots \right)_{J_1 \dots J_4}^{I_1 \dots I_4}, \\ R^{I_1 \dots I_4}_{J_1 \dots J_4} &= \varepsilon^{I_1 \dots I_4 K_1 K_2 K_3 J} \varphi_{K_1 K_2 K_3 I} \left(\frac{1}{72} \delta_J^I \varphi_{J_1 \dots J_4} - \frac{1}{9} \delta_{[J_1}^I \varphi_{J|J_2 J_3 J_4]} \right), \\ \gamma^I{}_{J|J_1 \dots J_4} &= \left(1 + \frac{1}{3!} Q + \frac{1}{5!} Q^2 + \frac{1}{7!} Q^3 + \dots \right)_{J|K}^I{}^L \left(\frac{4}{3} \delta_{[J_1}^K \varphi_{L|J_2 J_3 J_4]} - \frac{1}{6} \delta_L^K \varphi_{J_1 \dots J_4} \right), \\ \gamma^{I_1 \dots I_4}{}_I{}^J &= -\frac{1}{12} \left(1 + \frac{1}{3!} R + \frac{1}{5!} R^2 + \frac{1}{7!} R^3 + \dots \right)_{J_1 \dots J_4}^{I_1 \dots I_4} \varepsilon^{J_1 \dots J_4 K_1 K_2 K_3 J} \varphi_{K_1 K_2 K_3 I}. \quad (3.4.11) \end{aligned}$$

Conjugation with level 1 and level 2 elements is performed by Taylor-expanding the exponents. The generalised vielbein is

$$E_\Pi{}^A = (\det e)^{-\frac{1}{2}} \times$$

$$\begin{pmatrix} e_\mu^a & \alpha_{\mu|J_1J_2} & \alpha_\mu^{J_1J_2} & \alpha_{\mu|a} & \alpha_{\mu|aK}^L & \alpha_{\mu|aI_1\dots I_4} \\ 0 & \beta_{J_1J_2}^{I_1I_2} & \beta^{I_1I_2|J_1J_2} & \beta^{I_1I_2}_a & \beta^{I_1I_2}_{aK}^L & \beta_{aJ_1\dots J_4}^{I_1I_2} \\ 0 & \beta_{I_1I_2|J_1J_2} & \beta_{I_1I_2}^{J_1J_2} & \beta_{I_1I_2|a} & \beta_{I_1I_2|aK}^L & \beta_{I_1I_2|aJ_1\dots J_4} \\ 0 & 0 & 0 & (e^{-1})_a^\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & (e^{-1})_a^\mu \gamma^i_{J|K}^L & \gamma^i_{J|J_1\dots J_4} \\ 0 & 0 & 0 & 0 & (e^{-1})_a^\mu \gamma^{I_1\dots I_4}_{K}^L & (e^{-1})_a^\mu \gamma^{I_1\dots I_4}_{J_1\dots J_4} \end{pmatrix} \quad (3.4.12)$$

The quantities in the above matrix which have world indices are given in terms of quantities with all tangent indices by

$$\alpha_{\mu|J_1J_2} = e_\mu^b \alpha_{b|J_1J_2}, \quad \alpha_\mu^{J_1J_2} = e_\mu^b \alpha_b^{J_1J_2}, \text{ etc}$$

as well as

$$\begin{aligned} \beta_{J_1J_2}^{I_1I_2} &= (f^{-1})^{I_1I_2}_{K_1K_2} \beta_{J_1J_2}^{K_1K_2}, \quad \beta_{I_1I_2|J_1J_2} = f^{K_1K_2}_{I_1I_2} \beta_{K_1K_2|J_1J_2}, \\ \beta_{I_1I_2|J_1J_2} &= (f^{-1})^{I_1I_2}_{K_1K_2} \beta^{K_1K_2|J_1J_2} \beta_{I_1I_2}^{J_1J_2} = f^{K_1K_2}_{I_1I_2} \beta_{K_1K_2}^{J_1J_2}, \end{aligned}$$

which form the generalised vielbein on the coset space of the non-linear realisation of $E_7 \otimes_s l^{56}$ with local subgroup $SU(8)$, and in addition

$$\begin{aligned} \beta^{I_1I_2}_a &= (f^{-1})^{I_1I_2}_{K_1K_2} \beta^{K_1K_2}_a, \quad \beta_{I_1I_2|a} = f^{K_1K_2}_{I_1I_2} \beta_{K_1K_2|a} \\ \beta^{KI_1KI_2}_{aK}^L &= (f^{-1})^{I_1I_2}_{K_1K_2} \beta^{K_1K_2}_{aK}^L, \quad \beta_{aJ_1\dots J_4}^{I_1I_2} = (f^{-1})^{I_1I_2}_{K_1K_2} \beta_{aJ_1\dots J_4}^{K_1K_2} \\ \beta_{I_1I_2|aK}^L &= f^{K_1K_2}_{I_1I_2} \beta_{K_1K_2|aK}^L, \quad \beta_{I_1I_2|aJ_1\dots J_4} = f^{K_1K_2}_{I_1I_2} \beta_{K_1K_2|aJ_1\dots J_4}, \\ \gamma^i_{J|K}^L &= (f^{-1})^i_M f^N_J \gamma^M_{N|K}^L, \quad \gamma^i_{J|J_1\dots J_4} = (f^{-1})^i_M f^N_J \gamma^M_{N|J_1\dots J_4} \\ \gamma^{I_1\dots K=I_4}_{K}^L &= (f^{-1})^{I_1\dots I_4}_{K_1\dots K_4} \gamma^{K_1\dots K_4}_K^L, \quad \gamma^{I_1\dots I_4}_{J_1\dots J_4} = (f^{-1})^{I_1\dots I_4}_{K_1\dots K_4} \gamma^{K_1\dots K_4}_{J_1\dots J_4} \end{aligned}$$

With these definitions the symbols in the first line of the matrix are given by

$$\begin{aligned} \alpha_{a|I_1I_2} &= -A_{aI_1I_2}, \quad \alpha_a^{I_1I_2} = -A_a^{I_1I_2}, \quad \alpha_{a|b} = -\hat{h}_{(ab)} - \frac{1}{2} A_{[aI_1I_2} A_{b]}^{I_1I_2}, \\ \alpha_{a|bI}^J &= \frac{1}{2} A_{abI}^J + \frac{1}{2} A_{(aKI} A_{b)}^{KJ}, \quad \alpha_{a|bI_1\dots I_4} = \frac{1}{6} A_{abI_1\dots I_4} \\ &\quad - \frac{1}{2} A_{a[I_1I_2} A_{bI_3I_4]} + \frac{1}{48} \varepsilon_{I_1\dots I_8} A_a^{I_5I_6} A_b^{I_7I_8}, \end{aligned} \quad (3.4.13)$$

and the second line by

$$\begin{aligned} \beta^{I_1I_2}_a &= \beta_{J_1J_2}^{I_1I_2} A_a^{J_1J_2} - \beta^{I_1I_2|J_1J_2} A_{aJ_1J_2}, \quad \beta^{I_1I_2}_{aI}^J = -\beta_{KI}^{I_1I_2} A_a^{KJ} - \beta^{I_1I_2|KJ} A_{aKI}, \\ \beta_{aJ_1\dots J_4}^{I_1I_2} &= \beta_{[J_1J_2}^{I_1I_2} A_{aJ_3J_4]} - \frac{1}{24} \varepsilon_{J_1\dots J_8} \beta^{I_1I_2|J_5J_6} A_a^{J_7J_8}, \end{aligned}$$

$$\begin{aligned}\beta_{I_1 I_2 | a} &= -\beta_{I_1 I_2}^{J_1 J_2} A_{a J_1 J_2} + \beta_{I_1 I_2 | J_1 J_2} A_a^{J_1 J_2}, \quad \beta_{I_1 I_2 | a I}^J = -\beta_{I_1 I_2}^{K J} A_{a K I} - \beta_{I_1 I_2 | K I} A_a^{K J}, \\ \beta_{I_1 I_2 | a J_1 \dots J_4} &= -\frac{1}{24} \varepsilon_{J_1 \dots J_8} \beta_{I_1 I_2}^{J_5 J_6} A_a^{J_7 J_8} + \beta_{I_1 I_2 | [J_1 J_2} A_{a J_3 J_4]}.\end{aligned}\tag{3.4.14}$$

4 The non-linear realisation of A_1^{+++} and its generalised vielbein

As we have mentioned the non-linear realisations of the semi-direct product of very extended A_1 , denoted A_1^{+++} with its first fundamental representation, denoted l_1 is conjectured to lead to the complete low energy effective action for four dimensional gravity [7]. The Dynkin diagram for the Kac-Moody algebra A_1^{+++} is

$$\begin{array}{ccccccc} \bullet & - & \bullet & - & \bullet & = & \otimes \\ 1 & & 2 & & 3 & & 4 \end{array}$$

which corresponds to the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}.\tag{4.1}$$

The four dimensional theory appears when we delete node four, as indicated in the above diagram, to leave the algebra $GL(4)$. Decomposing A_1^{+++} into this subalgebra we find that the positive level generators of to level 2 are given by

$$K^a{}_b; R^{ab}; R^{ab,cd},\tag{4.2}$$

where the generators obey the conditions $R^{ab} = R^{ab}$ and $R^{ab,cd} = R^{ab,(cd)} = R^{[ab],cd}$, while the negative level generators are

$$R_{ab}; R_{ab,cd}\tag{4.3}$$

and satisfy similar constraints. The level ± 2 generators satisfy the conditions

$$R^{[ab,c]d} = R_{[ab,c]d} = 0.\tag{4.4}$$

For this Kac-Moody algebra the level is the number of up minus down $GL(4)$ indices on the generator divided by two.

The A_1^{+++} algebra can be constructed in the usual way, see reference [8] for a review of this process in the context of E_{11} . The commutators for the listed generators preserve the level and must obey the Jacobi identities, as such one proceeds level by level writing down the most general right-hand side for each commutator and then tests the Jacobi identities level by level. The generators belong to representations of $GL(4)$ and so their commutators with the generators $K^a{}_b$ are

$$[K^a{}_b, K^c{}_d] = \delta_b^c K^a{}_d - \delta_d^a K^c{}_b,$$

$$\begin{aligned}
[K^a_b, R^{a_1 a_2}] &= 2 \delta_b^{(a_1} R^{a|a_2)}, \quad [K^a_b, R_{a_1 a_2}] = -2 \delta_{(a_1}^a R_{b|a_2)}, \\
[K^a_b, R^{cd,ef}] &= \delta_b^c R^{ad,ef} + \delta_b^{a_2} R^{ca,ef} + \delta_b^e R^{cd,af} + \delta_b^f R^{cd,ea}, \\
[K^a_b, R_{cd,ef}] &= -\delta_c^a R_{bd,ef} - \delta_d^a R_{cb,ef} - \delta_e^a R_{cd,bf} - \delta_f^a R_{cd,eb}.
\end{aligned} \tag{4.5}$$

The level 2 (-2) two commutators must give on the right-hand side the unique level 2 (-2) generators and can be written in the form

$$[R^{ab}, R^{cd}] = R^{ac,bd} + R^{bd,ac}, \quad [R_{ab}, R_{cd}] = R_{ac,bd} + R_{bd,ac}. \tag{4.6}$$

where the normalisation of the level 2 (-2) generators are fixed by these relations. The reader may verify that the right-hand side of these commutators do indeed have the symmetries of the generators which occur in the left-hand side using the constraints on the generators given below equation (4.2). The commutators between the positive and negative level generators are given by

$$\begin{aligned}
[R^{ab}, R_{cd}] &= 2 \delta_c^{(a} K^{b)}_d - \delta_{cd}^{(ab)} \sum_e K^e_e, \\
[R^{ab,cd}, R_{ef}] &= \delta_{ef}^{(bd)} R^{ac} + \delta_{ef}^{(bc)} R^{ad} - \delta_{ef}^{(ac)} R^{bd} - \delta_{ef}^{(ad)} R^{bc}, \\
[R_{ab,cd}, R^{ef}] &= \delta_{bd}^{ef} R_{ac} + \delta_{bc}^{ef} R_{ad} - \delta_{ac}^{ef} R_{bd} - \delta_{ad}^{ef} R_{bc}.
\end{aligned} \tag{4.7}$$

where $\delta_{cd}^{(ab)} = \delta_c^{(a} \delta_d^{b)}$.

The relation of the above generators to the Chevalley generators of A_1^{+++} is given by

$$\begin{aligned}
H_1 &= K^1_1 - K^2_2, \quad H_2 = K^2_2 - K^3_3, \quad H_3 = K^3_3 - K^4_4 \\
H_4 &= -K^1_1 - K^2_2 - K^3_3 + K^4_4.
\end{aligned} \tag{4.8}$$

$$E_1 = K^1_2, \quad E_2 = K^2_3, \quad E_3 = K^3_4, \quad E_4 = R^{44}, \tag{4.9}$$

$$F_1 = K^2_1, \quad F_2 = K^3_2, \quad F_3 = K^4_3, \quad F_4 = R_{44} \tag{4.10}$$

One can verify that they satisfy the defining relations

$$[H_a, E_b] = A_{ab} E_b, \quad [E_a, F_b] = \delta_{ab} H_a, \quad [H_a, F_b] = -A_{ab} F_b \tag{4.11}$$

where A_{ab} is the Cartan matrix of A_1^{+++} given in equation (4.1).

The Cartan involution acts on the generators of A_1^{+++} as follows

$$I_c(K^a_b) = -K^b_a, \quad I_c(R_{ab}) = -R^{ab}, \quad I_c(R^{ab,cd}) = R_{ab,cd}, \tag{4.12}$$

The reader may verify that it leaves invariant the above commutators.

We pause here to review how the above construction of the A_1^{+++} algebra was carried out as this can act as an illustration of how to construct any Kac-Moody algebra from a knowledge of the generators. We have first written down the commutators of the known generators of equations (4.2) and (4.3) which are consistent with the level, $SL(4)$ algebra,

the Cartan involution and the symmetries of the indices on the generators. Strictly we should have included arbitrary constants on the right-hand sides of these commutators, that is, two constants in first of equations (4.7) and one constant in the last two of the equations (4.7), which are related by the action of the Cartan involution. The Jacobi identity $[[R^{ab}, R^{cd}], R_{ef}] + \dots = 0$ then gives one relation between these three constants.

We have then consider the Chevalley relations which by definition must satisfy the relations of equation (4.11). Those for the first three nodes, that is, E_a, F_a and H_a , $a = 1, 2, 3$ are just those for the subalgebra A_3 and are given in equation (4.8-10). The Chevalley generators E_4 must be constructed out of the level one generators R^{ab} . It must also commute with F_1, F_2 and F_3 and as a result it must, up to scale, be R^{44} . We can choose it to be $E_4 = R^{44}$. Similarly, or using the Cartan involution, we find that $F_4 = R_{44}$. The Chevalley generator H_4 must be a sum of the K^a_a generators and finding the correct relations with E_1, \dots, E_4 we find it is as given in equation (4.8). Finally, we impose that $[E_4, F_a] = 2H_4 = [R^{44}, R_{44}]$ which using the first of equations (4.7) fixes the two constants we should have introduced in this relations to be as they are given.

The l_1 representation generators up to level two are given by

$$P_a; Z^a; Z^{abc}, Z^{ab,c}, \quad (4.13)$$

where $Z^{abc} = Z^{(abc)}$, $Z^{ab,c} = Z^{[ab],c}$ and $Z^{[ab,c]} = 0$. Their commutators with the level 0 generators of $GL(4)$ are given by

$$\begin{aligned} [K^a_b, P_c] &= -\delta^a_c P_b + \frac{1}{2} \delta^a_b P_c, \quad [K^a_b, Z^c] = \delta^c_b Z^a + \frac{1}{2} \delta^a_b Z^c, \\ [K^a_b, Z^{cde}] &= \delta^c_b Z^{ade} + \delta^d_b Z^{cae} + \delta^e_b Z^{cda} + \frac{1}{2} \delta^a_b Z^{cde}, \\ [K^a_b, Z^{cd,e}] &= \delta^c_b Z^{ad,e} + \delta^d_b Z^{ca,e} + \delta^e_b Z^{cd,a} + \frac{1}{2} \delta^a_b Z^{cd,e}. \end{aligned} \quad (4.14)$$

The commutators of the level one A_1^{+++} generators with the l_1 generators must increase their level by one and they can be chosen to be of the form

$$[R^{ab}, P_c] = \delta^{(a}_c Z^{b)}, \quad [R^{ab}, Z^c] = Z^{abc} + Z^{c(a,b)}. \quad (4.15)$$

Using the Jacobi identities, the commutator of P_a with the level 2 generator of A_1^{+++} is found to be

$$[R^{ab,cd}, P_e] = -\delta^{[a}_e Z^{b]cd} + \frac{1}{4} \left(\delta^a_e Z^{b(c,d)} - \delta^b_e Z^{a(c,d)} \right) - \frac{3}{8} \left(\delta^c_e Z^{ab,d} + \delta^d_e Z^{ab,c} \right). \quad (4.16)$$

The commutators with level-lowering generators are given by

$$\begin{aligned} [R_{ab}, P_c] &= 0, \quad [R_{ab}, Z^c] = 2\delta^c_{(a} P_{b)}, \\ [R_{ab}, Z^{cde}] &= \frac{2}{3} \left(\delta^{cd}_{(ab)} Z^e + \delta^{de}_{(ab)} Z^c + \delta^{ec}_{(ab)} Z^d \right), \end{aligned}$$

$$[R_{ab}, Z^{cd,e}] = \frac{4}{3} \left(\delta_{(ab)}^{de} Z^c - \delta_{(ab)}^{ce} Z^d \right). \quad (4.17)$$

The very first relation reflects the fact that the l_1 representation is a lowest weight representation.

Having constructed the $A_1^{+++} \otimes_s l_1$ algebra up to level two we can construction its non-linear realisation. The group element $g = g_l g_A$ can, up to level two, be written in the form

$$\begin{aligned} g_l &= \exp \left(x^a P_a + y_a Z^a + x_{abc} Z^{abc} + x_{ab,c} Z^{ab,c} \right), \\ g_A &= \exp \left(h_a{}^b K^a{}_b \right) \exp \left(A_{ab,cd} R^{ab,cd} \right) \exp \left(A_{ab} R^{ab} \right), \end{aligned} \quad (4.18)$$

We find that we have introduced the fields

$$h_a{}^b; A_{ab}; A_{ab,cd} \quad (4.19)$$

where $A_{ab} = A_{(ab)}$; $A_{ab,cd} = A_{[ab],cd} = A_{ab,(cd)}$, and the coordinates

$$x^a; y_a; x_{abc}, x_{ab,c}, \quad (4.20)$$

where $x_{abc} = x_{(abc)}$, $x_{ab,c} = x_{[ab],c}$. The field $h_a{}^b$ is the usual graviton while the field A_{ab} is the dual graviton. Analogously the coordinates x^a are the usual coordinates of space-time while the coordinates y_a are the dual coordinates.

This non-linear realisation is a good arena in which to discuss the dual graviton and the resulting dynamics will be discussed elsewhere. Here we will content ourselves with calculating the generalised vielbein up to level two. We will use the definition of equation (1.10) which involves conjugating the l_1 generators with g_A using the above algebra. Conjugation with level 0 group element gives

$$\begin{aligned} &\exp \left(-h_a{}^b K^a{}_b \right) \left\{ P_\mu, Z^\mu, Z^{\mu_1\mu_2\mu_3}, Z^{\mu_1\mu_2,\mu_3} \right\} \exp \left(h_a{}^b K^a{}_b \right) = \\ &= (\det e)^{-\frac{1}{2}} \left\{ e_\mu{}^a P_a, (e^{-1})_a{}^\mu Z^a, (e^{-1})_{(a_1 a_2 a_3)}^{(\mu_1 \mu_2 \mu_3)} Z^{a_1 a_2 a_3}, (e^{-1})_{[a_1 a_2], a_3}^{[\mu_1 \mu_2], \mu_3} Z^{a_1 a_2, a_3} \right\}, \end{aligned} \quad (4.21)$$

where $e_a{}^b = (e^h)_a{}^b$ and

$$\begin{aligned} (e^{-1})_{a_1 \dots a_n}^{\mu_1 \dots \mu_n} &= (e^{-1})_{[a_1}^{\mu_1} \dots (e^{-1})_{a_n]}^{\mu_n}, \\ (e^{-1})_{[a_1 a_2], a_3}^{[\mu_1 \mu_2], \mu_3} &= (e^{-1})_{[a_1}^{\mu_1} (e^{-1})_{a_2]}^{\mu_2} (e^{-1})_{a_3}^{\mu_3} - (e^{-1})_{[a_1}^{\mu_1} (e^{-1})_{a_2}^{\mu_2} (e^{-1})_{a_3]}^{\mu_3}. \end{aligned} \quad (4.22)$$

Conjugating with positive level generators can be obtained by Taylor-expanding the exponents and truncating the series by level 2. For the E_{11} level one generators we have

$$\begin{aligned} &\exp \left(-A_{bc} R^{bc} \right) \left\{ P_a, Z^a \right\} \exp \left(A_{bc} R^{bc} \right) = \\ &= \left\{ P_a - A_{ab} Z^b + \frac{1}{2} A_{ab} A_{cd} Z^{bcd} + \frac{1}{2} A_{ab} A_{cd} Z^{bc,d}, Z^a - A_{bc} Z^{abc} - A_{bc} Z^{ab,c} \right\}. \end{aligned} \quad (4.23)$$

while for the E_{11} level 2 generator:

$$\begin{aligned} & \exp(A_{bc,de} R^{bc,de}) P_a \exp(A_{bc,de} R^{bc,de}) \\ &= P_a + A_{ab,cd} Z^{bcd} + \left(\frac{3}{4} A_{bc,ad} - \frac{1}{2} A_{ab,cd} \right) Z^{bc,d}. \end{aligned} \quad (4.24)$$

As we are only computing up to level two, that is, up to the l_1 elements Z^{abc} and $Z^{ab,c}$ the order in which we calculate the action of the group elements on the l_1 generators is irrelevant. Combining these results together we find that the generalised vielbein up to level two is given by

$$\begin{aligned} E_{\Pi}^A &= (\det e)^{-\frac{1}{2}} \\ &= \begin{pmatrix} e_{\mu}^a & e_{\mu}^b \alpha_{b|a} & e_{\mu}^b \alpha_{b|a_1 a_2 a_3} & e_{\mu}^b \alpha_{b|a_1 a_2, a_3} \\ 0 & (e^{-1})_a^{\mu} & (e^{-1})_b^{\mu} \beta_{a_1 a_2 a_3}^b & (e^{-1})_b^{\mu} \beta_{a_1 a_2, a_3}^b \\ 0 & 0 & (e^{-1})_{(a_1 a_2 a_3)}^{(\mu_1 \mu_2 \mu_3)} & 0 \\ 0 & 0 & 0 & (e^{-1})_{[a_1 a_2], a_3}^{[\mu_1 \mu_2], \mu_3} \end{pmatrix}, \end{aligned} \quad (4.25)$$

where the symbols in the first line are given by

$$\begin{aligned} \alpha_{a|b} &= -A_{ab}, \quad \alpha_{a|a_1 a_2 a_3} = \alpha_{a|(a_1 a_2 a_3)} = A_{a(a_1, a_2 a_3)} + \frac{1}{2} A_{a(a_1} A_{a_2 a_3)}, \\ \alpha_{a|a_1 a_2, a_3} &= \alpha_{a|[a_1 a_2], a_3} = \frac{3}{4} A_{a_1 a_2, a_3 a} - \frac{1}{2} A_{a[a_1, a_2] a_3} + \frac{1}{2} A_{a[a_1} A_{a_2] a_3}, \end{aligned} \quad (4.26)$$

while the symbols in the second line are given by

$$\beta_{a_1 a_2 a_3}^a = \beta_{(a_1 a_2 a_3)}^a = -\delta_{(a_1}^a A_{a_2 a_3)}, \quad \beta_{a_1 a_2, a_3}^a = \beta_{[a_1 a_2], a_3}^a = -\delta_{[a_1}^a A_{a_2] a_3}. \quad (4.27)$$

5 Conclusion

In this paper we have reviewed how to construct the generalised vielbein associated with the generalised space-time that arises in the non-linear realisation of $E_{11} \otimes_s l_1$. We find the generalised vielbein up to, and including, the level containing the dual graviton in eleven, five and four dimensions as well as for the ten dimensional IIB theory. To find these results one requires $E_{11} \otimes_s l_1$ algebra up to the level concerned. These algebras were previously known in eleven and four dimensions and in this paper we have also found them in five dimensions and for the ten dimensional IIB theory, the explicit formulae being given in appendix A.

In a recent paper the gauge transformations of the fields in the $E_{11} \otimes_s l_1$ non-linear realisation were proposed [29]. These are formulated in terms of the generalised vielbein and the results for this object given in this paper will prove useful for finding the explicit gauge transformations.

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Appendix A

For convenience we give in this appendix the $E_{11} \otimes_s l_1$ algebra appropriate to four, five and eleven dimensions and also for the IIB ten dimensional theory.

A.1 $D = 11$ algebra

In this appendix we repeat, for convenience, the $E_{11} \otimes_s l_1$ algebra decomposed into representations of $GL(11)$ [11]. The commutators of the E_{11} generators with the generators of K^a_b are given by

$$\begin{aligned}
[K^a_b, K^c_d] &= \delta_b^c K^a_d - \delta_d^a K^c_b, \quad [K^a_b, R^{a_1 a_2 a_3}] = 3 \delta_b^{[a_1} R^{a|a_2 a_3]}, \\
[K^a_b, R_{a_1 a_2 a_3}] &= -3 \delta_{[a_1}^a R_{b|a_2 a_3]}, \\
[K^a_b, R^{a_1 \dots a_5}] &= 5 \delta_b^{[a_1} R^{a|a_2 \dots a_5]}, \quad [K^a_b, R_{a_1 \dots a_5}] = -5 \delta_{[a_1}^a R_{b|a_2 \dots a_5]}, \\
[K^a_b, R^{a_1 \dots a_8}] &= 8 \delta_b^{[a_1} R^{a|a_2 \dots a_8]}, \quad [K^a_b, R_{a_1 \dots a_8}] = -8 \delta_{[a_1}^a R_{b|a_2 \dots a_8]}, \\
[K^a_b, R^{a_1 \dots a_7, c}] &= 7 \delta_b^{[a_1} R^{a|a_2 \dots a_7], c} + \delta_b^c R^{a_1 \dots a_7, a}, \\
[K^a_b, R_{a_1 \dots a_7, c}] &= -7 \delta_{[a_1}^a R_{b|a_2 \dots a_7], c} - \delta_c^a R_{a_1 \dots a_7, b}.
\end{aligned} \tag{A.1.1}$$

The positive level commutators are given by

$$[R^{a_1 a_2 a_3}, R^{a_4 a_5 a_6}] = 2 R^{a_1 \dots a_6}, \quad [R^{a_1 a_2 a_3}, R^{b_1 \dots b_6}] = 6 R^{a_1 a_2 a_3 [b_1 \dots b_5, b_6]},$$

while the negative level commutators are given by

$$[R_{a_1 a_2 a_3}, R_{a_4 a_5 a_6}] = 2 R_{a_1 \dots a_6}, \quad [R_{a_1 a_2 a_3}, R_{b_1 \dots b_6}] = 6 R_{a_1 a_2 a_3 [b_1 \dots b_5, b_6]}. \tag{A.1.2}$$

The commutators between the positive and negative level generators are given by

$$\begin{aligned}
[R^{a_1 a_2 a_3}, R_{b_1 b_2 b_3}] &= 18 \delta_{[b_1 b_2}^{[a_1 a_2} K^{a_3]}_{b_3]} - 2 \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} K^a_a, \\
[R^{a_1 a_2 a_3}, R_{b_1 \dots b_6}] &= 60 \delta_{[b_1 b_2 b_3}^{a_1 a_2 a_3} R_{b_4 b_5 b_6]}, \\
[R^{a_1 a_2 a_3}, R_{b_1 \dots b_8, b}] &= 112 \delta_{[b_1 b_2 b_3}^{a_1 a_2 a_3} R_{b_4 \dots b_8] b} - 112 \delta_{[b_1 b_2 | b]}^{a_1 a_2 a_3} R_{b_3 \dots b_8]}, \\
[R_{a_1 a_2 a_3}, R^{b_1 \dots b_6}] &= 60 \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} R^{b_4 b_5 b_6]}, \\
[R_{a_1 a_2 a_3}, R^{b_1 \dots b_8, b}] &= 112 \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} R^{b_4 \dots b_8] b} - 112 \delta_{a_1 a_2 a_3}^{[b_1 b_2 | b]} R^{b_3 \dots b_8]}, \\
[R^{a_1 \dots a_6}, R_{b_1 \dots b_6}] &= -1080 \delta_{[b_1 \dots b_5}^{[a_1 \dots a_5} K^{a_6]}_{b_6]} + 120 \delta_{b_1 \dots b_6}^{a_1 \dots a_6} K^a_a, \\
[R_{a_1 \dots a_6}, R^{b_1 \dots b_8, b}] &= -3360 \delta_{a_1 \dots a_6}^{[b_1 \dots b_6} R^{b_7 b_8] b} - 3360 \delta_{a_1 \dots a_6}^{[b_1 \dots b_5 | b]} R^{b_6 b_7 b_8]},
\end{aligned}$$

$$[R^{a_1 \dots a_6}, R_{b_1 \dots b_8, b}] = -3360 \delta_{[b_1 \dots b_6]}^{a_1 \dots a_6} R_{b_7 b_8] b} - 3360 \delta_{[b_1 \dots b_5 | b]}^{a_1 \dots a_6} R_{b_6 b_7 b_8]}, \quad (A.1.3)$$

The commutators of the $GL(11)$ generators with those of the l_1 representation are given by [6]

$$\begin{aligned} [K^a_b, P_c] &= -\delta_c^a P_b + \frac{1}{2} \delta_b^a P_c, \quad [K^a_b, Z^{a_1 a_2}] = 2 \delta_b^{[a_1} Z^{a] a_2]} + \frac{1}{2} \delta_b^a Z^{a_1 a_2}, \\ [K^a_b, Z^{a_1 \dots a_5}] &= 5 \delta_b^{[a_1} Z^{a] a_2 \dots a_5]} + \frac{1}{2} \delta_b^a Z^{a_1 \dots a_5}, \\ [K^a_b, Z^{a_1 \dots a_8}] &= 8 \delta_b^{[a_1} Z^{a] a_2 \dots a_8]} + \frac{1}{2} \delta_b^a Z^{a_1 \dots a_8}, \\ [K^a_b, Z^{a_1 \dots a_7, c}] &= 7 \delta_b^{[a_1} Z^{a] a_2 \dots a_7], c} + \delta_b^c Z^{a_1 \dots a_7, a} + \frac{1}{2} \delta_b^a Z^{a_1 \dots a_7, c}, \end{aligned} \quad (A.1.4)$$

The commutators of the positive root generators of E_{11} with l_1 generators are given by

$$\begin{aligned} [R^{a_1 a_2 a_3}, P_a] &= 3 \delta_a^{[a_1} Z^{a_2 a_3]}, \quad [R^{a_1 a_2 a_3}, Z^{a_4 a_5}] = Z^{a_1 \dots a_5}, \\ [R^{a_1 a_2 a_3}, Z^{b_1 \dots b_5}] &= Z^{b_1 \dots b_5 a_1 a_2 a_3} + Z^{b_1 \dots b_5 [a_1 a_2, a_3]} \\ [R^{a_1 \dots a_6}, P_a] &= -3 \delta_a^{[a_1} Z^{a_2 \dots a_6]}, \quad [R^{a_1 \dots a_6}, Z^{b_1 b_2}] = -Z^{b_1 b_2 a_1 \dots a_6} - Z^{b_1 b_2 [a_1 \dots a_5, a_6]}, \\ [R^{a_1 \dots a_8, a}, P_b] &= -\frac{4}{3} \delta_b^a Z^{a_1 \dots a_8} + \frac{4}{3} \delta_b^{[a_1} Z^{a_2 \dots a_8] a} + \frac{4}{3} \delta_b^{[a_1} Z^{a_2 \dots a_8], a}. \end{aligned} \quad (A.1.5)$$

While the commutators of the l_1 generators with the level minus one E_{11} generators are given by

$$\begin{aligned} [R_{a_1 a_2 a_3}, P_a] &= 0, \quad [R_{a_1 a_2 a_3}, Z^{b_1 b_2}] = 6 \delta_{[a_1 a_2}^{b_1 b_2} P_{a_3]}, \\ [R_{a_1 a_2 a_3}, Z^{b_1 \dots b_5}] &= 60 \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} Z^{b_4 b_5]}, \quad [R_{a_1 a_2 a_3}, Z^{b_1 \dots b_8}] = -42 \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} Z^{b_4 \dots b_8]}, \\ [R_{a_1 a_2 a_3}, Z^{b_1 \dots b_7, b}] &= \frac{945}{4} \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} Z^{b_4 \dots b_7] b} + \frac{945}{4} \delta_{a_1 a_2 a_3}^{[b_1 b_2 | b]} Z^{b_3 \dots b_7}. \end{aligned} \quad (A.1.6)$$

A.2 $D = 10$ algebra

In this appendix we give the commutators of $E_{11} \otimes_s l_1$ algebra, decomposed into representations of $GL(10) \otimes SL(2, R)$. Parts of this algebra for the form generators were given in references [12] and [27]. The l_1 multiplet and their commutators with the E_{11} generators are given for the first time in this paper as are many of the commutators of the E_{11} algebra that involve the negative level generators. The commutators of the E_{11} generators with the $SL(10)$ generators K^a_b are

$$\begin{aligned} [K^a_b, K^c_d] &= \delta_b^c K^a_d - \delta_d^a K^c_b, \quad [K^a_b, R_{\alpha\beta}] = 0, \\ [K^a_b, R_{\alpha}^{a_1 a_2}] &= 2 \delta_b^{[a_1} R_{\alpha}^{a] a_2]}, \quad [K^a_b, R_{a_1 a_2}^{\alpha}] = -2 \delta_{[a_1}^a R_{b] a_2}^{\alpha}, \\ [K^a_b, R^{a_1 \dots a_4}] &= 4 \delta_b^{[a_1} R^{a] a_2 a_3 a_4]}, \quad [K^a_b, R_{a_1 \dots a_4}] = -4 \delta_{[a_1}^a R_{b] a_2 a_3 a_4]}, \end{aligned}$$

$$\begin{aligned}
[K^a_b, R^{a_1 \dots a_6}_\alpha] &= 6 \delta_b^{[a_1} R^{a|a_2 \dots a_6]}_\alpha, \quad [K^a_b, R^{a_1 \dots a_6}_\alpha] = -6 \delta_{[a_1}^a R^\alpha_{|b|a_2 \dots a_6]}, \\
[K^a_b, R^{a_1 \dots a_8}_{\alpha\beta}] &= 8 \delta_b^{[a_1} R^{a|a_2 \dots a_8]}_{\alpha\beta}, \quad [K^a_b, R^{\alpha\beta}_{a_1 \dots a_8}] = -8 \delta_{[a_1}^a R^{\alpha\beta}_{|b|a_2 \dots a_8]}, \\
[K^a_b, R^{a_1 \dots a_7, c}] &= 7 \delta_b^{[a_1} R^{a|a_2 \dots a_7], c} + \delta_b^c R^{a_1 \dots a_7, a}, \\
[K^a_b, R_{a_1 \dots a_7, c}] &= -7 \delta_{[a_1}^a R_{|b|a_2 \dots a_7], c} - \delta_c^a R_{a_1 \dots a_7, b}.
\end{aligned} \tag{A.2.1}$$

The commutators of the E_{11} generators with the $SL(2, R)$ generators $R_{\alpha\beta}$ are

$$\begin{aligned}
[R_{\alpha\beta}, R_{\gamma\delta}] &= \delta_{(\alpha}^\sigma \varepsilon_{\beta)\gamma} R_{\sigma\delta} + \delta_{(\alpha}^\sigma \varepsilon_{\beta)\delta} R_{\gamma\sigma}, \\
[R_{\alpha\beta}, R^{a_1 a_2}_\gamma] &= \delta_{(\alpha}^\delta \varepsilon_{\beta)\gamma} R_\delta^{a_1 a_2}, \quad [R_{\alpha\beta}, R^{a_1 a_2}_\gamma] = -\delta_{(\alpha}^\gamma \varepsilon_{\beta)\delta} R_\delta^{a_1 a_2}, \\
[R_{\alpha\beta}, R^{a_1 \dots a_4}] &= 0, \quad [R_{\alpha\beta}, R_{a_1 \dots a_4}] = 0, \\
[R_{\alpha\beta}, R^{a_1 \dots a_6}_\gamma] &= \delta_{(\alpha}^\delta \varepsilon_{\beta)\gamma} R_\delta^{a_1 \dots a_6}, \quad [R_{\alpha\beta}, R^{a_1 \dots a_6}_\gamma] = -\delta_{(\alpha}^\gamma \varepsilon_{\beta)\delta} R_\delta^{a_1 \dots a_6}, \\
[R_{\alpha\beta}, R^{a_1 \dots a_8}_{\gamma\delta}] &= \delta_{(\alpha}^\sigma \varepsilon_{\beta)\gamma} R_{\sigma\delta}^{a_1 \dots a_8} + \delta_{(\alpha}^\sigma \varepsilon_{\beta)\delta} R_{\gamma\sigma}^{a_1 \dots a_8}, \\
[R_{\alpha\beta}, R^{a_1 \dots a_8}_{\gamma\delta}] &= -\delta_{(\alpha}^\gamma \varepsilon_{\beta)\sigma} R_{a_1 \dots a_8}^{\sigma\delta} - \delta_{(\alpha}^\delta \varepsilon_{\beta)\sigma} R_{a_1 \dots a_8}^{\gamma\sigma}, \\
[R_{\alpha\beta}, R^{a_1 \dots a_7, b}] &= 0, \quad [R_{\alpha\beta}, R_{a_1 \dots a_7, b}] = 0.
\end{aligned} \tag{A.2.2}$$

The commutators of the positive level E_{11} generators are given by

$$\begin{aligned}
[R^{a_1 a_2}_\alpha, R^{a_3 a_4}_\beta] &= -\varepsilon_{\alpha\beta} R^{a_1 \dots a_4}, \quad [R^{a_1 a_2}_\alpha, R^{a_3 a_4}_\beta] = -\varepsilon^{\alpha\beta} R_{a_1 \dots a_4}, \\
[R^{a_1 a_2}_\alpha, R^{a_3 \dots a_6}] &= 4 R^{a_1 \dots a_6}_\alpha, \quad [R^{a_1 a_2}_\alpha, R_{a_3 \dots a_6}] = 4 R_{a_1 \dots a_6}^\alpha, \\
[R^{a_1 a_2}_\alpha, R^{a_3 \dots a_8}_\beta] &= -R^{a_1 \dots a_8}_{\alpha\beta} - \varepsilon_{\alpha\beta} R^{a_1 a_2 [a_3 \dots a_7, a_8]}, \\
[R^{a_1 a_2}_\alpha, R^{a_3 \dots a_8}_{\beta}] &= -R^{\alpha\beta}_{a_1 \dots a_8} - \varepsilon^{\alpha\beta} R_{a_1 a_2 [a_3 \dots a_7, a_8]},
\end{aligned} \tag{A.2.3}$$

while

$$[R^{a_1 \dots a_4}, R^{a_5 \dots a_8}] = \frac{8}{3} R^{a_1 \dots a_4 [a_5 a_6 a_7, a_8]}, \quad [R_{a_1 \dots a_4}, R_{a_5 \dots a_8}] = \frac{8}{3} R_{a_1 \dots a_4 [a_5 a_6 a_7, a_8]}. \tag{A.2.4}$$

To find the commutators between positive and negative level generators we need to utilize Jacobi identities. These commutators up to level 3 are given by

$$\begin{aligned}
[R^{a_1 a_2}_\alpha, R^{b_1 b_2}_\beta] &= 4 \delta_\alpha^\beta \delta_{[b_1}^{[a_1} K^{a_2]}_{b_2]} - \frac{1}{2} \delta_\alpha^\beta \delta_{b_1 b_2}^{a_1 a_2} K^d_d - 2 \delta_{b_1 b_2}^{a_1 a_2} \varepsilon^{\beta\gamma} R_{\alpha\gamma}, \\
[R^{a_1 a_2}_\alpha, R_{b_1 \dots b_4}] &= -12 \varepsilon_{\alpha\beta} \delta_{[b_1 b_2}^{a_1 a_2} R_{b_3 b_4]}^\beta, \quad [R^{a_1 a_2}_\alpha, R^{b_1 \dots b_4}] = -12 \varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} R_{b_3 b_4]}^{\beta], \\
[R^{a_1 \dots a_4}, R_{b_1 \dots b_4}] &= 12 \delta_{b_1 \dots b_4}^{a_1 \dots a_4} K^d_d - 96 \delta_{[b_1 b_2 b_3}^{[a_1 a_2 a_3} K^{a_4]}_{b_4]},
\end{aligned}$$

$$\begin{aligned}
[R_\alpha^{a_1 a_2}, R_{b_1 \dots b_6}^\beta] &= \frac{15}{2} \delta_\alpha^\beta \delta_{[b_1 b_2]}^{a_1 a_2} R_{b_3 \dots b_6}, \quad [R_{a_1 a_2}^\alpha, R_{\beta}^{b_1 \dots b_6}] = \frac{15}{2} \delta_\beta^\alpha \delta_{a_1 a_2}^{[b_1 b_2]} R^{b_3 \dots b_6}, \\
[R^{a_1 \dots a_4}, R_{b_1 \dots b_6}^\alpha] &= 90 \delta_{[b_1 \dots b_4]}^{a_1 \dots a_4} R_{b_5 b_6}^\alpha, \quad [R_{a_1 \dots a_4}, R_\alpha^{b_1 \dots b_4}] = 90 \delta_{a_1 \dots a_4}^{[b_1 \dots b_4]} R_\alpha^{b_5 b_6}, \\
[R_\alpha^{a_1 \dots a_6}, R_{b_1 \dots b_6}^\beta] &= 270 \delta_\alpha^\beta \delta_{[b_1 \dots b_5]}^{a_1 \dots a_5} K^{a_6}]_{b_6} - \frac{135}{4} \delta_\alpha^\beta \delta_{b_1 \dots b_6}^{a_1 \dots a_6} K^d_d - 45 \delta_{b_1 \dots b_6}^{a_1 \dots a_6} \varepsilon^{\beta \gamma} R_{\alpha \gamma}.
\end{aligned} \tag{A.2.5}$$

The commutators of level ∓ 4 generators with level ± 1 ones are

$$\begin{aligned}
[R_\alpha^{a_1 a_2}, R_{b_1 \dots b_8}^{\beta \gamma}] &= -56 \delta_\alpha^{(\beta} \delta_{[b_1 b_2]}^{a_1 a_2} R_{b_3 \dots b_8}^{\gamma)}, \quad [R_{a_1 a_2}^\alpha, R_{\beta \gamma}^{b_1 \dots b_8}] = -56 \delta_{(\beta}^\alpha \delta_{a_1 a_2}^{[b_1 b_2]} R_{\gamma)}^{b_3 \dots b_8}, \\
[R_\alpha^{a_1 a_2}, R_{b_1 \dots b_7, b}] &= -252 \varepsilon_{\alpha \beta} \delta_{[b_1 b_2]}^{a_1 a_2} R_{b_3 \dots b_7, b}^\beta + 252 \varepsilon_{\alpha \beta} \delta_{[b_1 b_2]}^{a_1 a_2} R_{b_3 \dots b_7 b}^\beta, \\
[R_{a_1 a_2}^\alpha, R^{b_1 \dots b_7, b}] &= -252 \varepsilon^{\alpha \beta} \delta_{a_1 a_2}^{[b_1 b_2]} R_{b_3 \dots b_7, b}^\beta + 252 \varepsilon^{\alpha \beta} \delta_{a_1 a_2}^{[b_1 b_2]} R_{b_3 \dots b_7 b}^\beta,
\end{aligned} \tag{A.2.6}$$

with levels ± 2 :

$$\begin{aligned}
[R^{a_1 \dots a_4}, R_{b_1 \dots b_8}^{\alpha \beta}] &= 0, \quad [R_{a_1 \dots a_4}, R_{\alpha \beta}^{b_1 \dots b_8}] = 0, \\
[R^{a_1 \dots a_4}, R_{b_1 \dots b_7, b}] &= -1260 \delta_{[b_1 \dots b_4]}^{a_1 \dots a_4} R_{b_5 b_6 b_7, b} + 1260 \delta_{[b_1 \dots b_4]}^{a_1 \dots a_4} R_{b_5 b_6 b_7 b}, \\
[R_{a_1 \dots a_4}, R^{b_1 \dots b_7, b}] &= -1260 \delta_{a_1 \dots a_4}^{[b_1 \dots b_4]} R^{b_5 b_6 b_7, b} + 1260 \delta_{a_1 \dots a_4}^{[b_1 \dots b_4]} R^{b_5 b_6 b_7 b},
\end{aligned} \tag{A.2.7}$$

with levels ± 3 :

$$\begin{aligned}
[R_\alpha^{a_1 \dots a_6}, R_{b_1 \dots b_8}^{\beta \gamma}] &= 1260 \delta_\alpha^{(\beta} \delta_{[b_1 \dots b_6]}^{a_1 \dots a_6} R_{b_7 b_8}^{\gamma)}, \quad [R_{a_1 \dots a_6}, R_{\beta \gamma}^{b_1 \dots b_8}] = 1260 \delta_{(\beta}^\alpha \delta_{a_1 \dots a_6}^{[b_1 \dots b_6]} R_{\gamma)}^{b_7 b_8}, \\
[R_\alpha^{a_1 \dots a_6}, R_{b_1 \dots b_7, b}] &= 1890 \varepsilon_{\alpha \beta} \delta_{[b_1 \dots b_6]}^{a_1 \dots a_6} R_{b_7, b}^\beta - 1890 \varepsilon_{\alpha \beta} \delta_{[b_1 \dots b_6]}^{a_1 \dots a_6} R_{b_7 b}^\beta, \\
[R_{a_1 \dots a_6}, R^{b_1 \dots b_7, b}] &= 1890 \varepsilon^{\alpha \beta} \delta_{a_1 \dots a_6}^{[b_1 \dots b_6]} R_{b_7, b}^\beta - 1890 \varepsilon^{\alpha \beta} \delta_{a_1 \dots a_6}^{[b_1 \dots b_6]} R_{b_7 b}^\beta,
\end{aligned} \tag{A.2.8}$$

and, finally, the commutators of level ± 4 generators between themselves are

$$\begin{aligned}
[R_{\alpha_1 \alpha_2}^{a_1 \dots a_8}, R_{b_1 \dots b_8}^{\beta_1 \beta_2}] &= -20160 \delta_{\alpha_1 \alpha_2}^{(\beta_1 \beta_2)} \delta_{[b_1 \dots b_7]}^{a_1 \dots a_7} K^{a_8}]_{b_8} \\
&+ 2520 \delta_{\alpha_1 \alpha_2}^{(\beta_1 \beta_2)} \delta_{b_1 \dots b_8}^{a_1 \dots a_8} K^d_d + 5040 \delta_{b_1 \dots b_8}^{a_1 \dots a_8} \delta_{(\alpha_1}^{(\beta_1} \varepsilon^{\beta_2) \gamma} R_{\alpha_2) \gamma}, \\
[R_{\alpha \beta}^{a_1 \dots a_8}, R_{b_1 \dots b_7, b}] &= 0, \quad [R_{a_1 \dots a_8}^{\alpha \beta}, R^{b_1 \dots b_7, b}] = 0, \\
[R^{a_1 \dots a_7, a}, R_{b_1 \dots b_7, b}] &= -11340 \delta_{b_1 \dots b_7}^{a_1 \dots a_7} K^a_b + 11340 \delta_{[b_1 \dots b_7]}^{a_1 \dots a_7} K^a_b] + 11340 \delta_{b_1 \dots b_7}^{[a_1 \dots a_7]} K^a_b \\
&- 79380 \delta_b^a \delta_{[b_1 \dots b_6]}^{[a_1 \dots a_6]} K^{a_7}]_{b_7} + 79380 \delta_{[b_1 \dots b_6]}^{[a_1 \dots a_6]} K^{a_7}]_{b_7} + 79380 \delta_{b_1 \dots b_6}^{[a_1 \dots a_6]} K^{a_7}]_{b_7} - \\
&- 90720 \delta_{[b_1 \dots b_7]}^{[a_1 \dots a_7]} K^a_b] + 11340 \delta_{b_1 \dots b_7}^{a_1 \dots a_7} \delta_b^a K^d_d - 11340 \delta_{b_1 \dots b_7}^{a_1 \dots a_7 a} K^d_d.
\end{aligned} \tag{A.2.9}$$

The action of the Cartan involution on the adjoint generators is given by

$$I_c(K^a_b) = -K^b_a, \quad I_c(R_{\alpha \beta}) = \varepsilon_{\alpha \gamma} \varepsilon_{\beta \delta} R_{\gamma \delta}, \quad I_c(R_\alpha^{a_1 a_2}) = -R_{a_1 a_2}^\alpha, \quad I_c(R^{a_1 \dots a_4}) = R_{a_1 \dots a_4},$$

$$I_c(R_\alpha^{a_1 \dots a_6}) = -R_{a_1 \dots a_6}^\alpha, \quad I_c(R_{\alpha_1 \alpha_2}^{a_1 \dots a_8}) = R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, \quad I_c(R^{a_1 \dots a_7, b}) = R_{a_1 \dots a_7, b}. \quad (A.2.10)$$

One can verify that the above commutators are preserve by this involution.

We now consider the commutators of the E_{11} generators with those of the l_1 representation. The members of the l_1 representation are most easily found using the Nutma programme Simplicie [30]. The commutators of the l_1 representation generators with the level 0 E_{11} generators, that is the $SL(11)$ generators, are given by

$$\begin{aligned} [K^a_b, P_c] &= -\delta_c^a P_b + \frac{1}{2} \delta_b^a P_c, \quad [K^a_b, Z_\alpha^c] = \delta_b^c Z_\alpha^a + \frac{1}{2} \delta_b^a Z_\alpha^c, \\ [K^a_b, Z^{a_1 a_2 a_3}] &= 3 \delta_b^{[a_1} Z^{a|a_2 a_3]} + \frac{1}{2} \delta_b^a Z^{a_1 a_2 a_3}, \quad [K^a_b, Z_\alpha^{a_1 \dots a_5}] \\ &= 5 \delta_b^{[a_1} Z_\alpha^{a|a_2 \dots a_5]} + \frac{1}{2} \delta_b^a Z_\alpha^{a_1 \dots a_5}, \\ [K^a_b, Z_{\alpha\beta}^{a_1 \dots a_7}] &= 7 \delta_b^{[a_1} Z_{\alpha\beta}^{a|a_2 \dots a_7]} + \frac{1}{2} \delta_b^a Z_{\alpha\beta}^{a_1 \dots a_7}, \\ [K^a_b, Z^{a_1 \dots a_7}] &= 7 \delta_b^{[a_1} Z^{a|a_2 \dots a_7]} + \frac{1}{2} \delta_b^a Z^{a_1 \dots a_7}, \\ [K^a_b, Z^{a_1 \dots a_6, c}] &= 6 \delta_b^{[a_1} Z^{a|a_2 \dots a_6], c} + \delta_b^c Z^{a_1 \dots a_6, a} + \frac{1}{2} \delta_b^a Z^{a_1 \dots a_6, c}. \end{aligned} \quad (A.2.11)$$

The commutators with the $SL(2)$ generators $R_{\alpha\beta}$ are

$$\begin{aligned} [R_{\alpha\beta}, P_a] &= 0, \quad [R_{\alpha\beta}, Z_\gamma^a] = \delta_{(\alpha}^\delta \varepsilon_{\beta)\gamma} Z_\delta^a, \\ [R_{\alpha\beta}, Z^{a_1 a_2 a_3}] &= 0, \quad [R_{\alpha\beta}, Z_\gamma^{a_1 \dots a_5}] = \delta_{(\alpha}^\delta \varepsilon_{\beta)\gamma} Z_\delta^{a_1 \dots a_5}, \\ [R_{\alpha\beta}, Z_{\gamma\delta}^{a_1 \dots a_7}] &= \delta_{(\alpha}^\sigma \varepsilon_{\beta)\gamma} Z_{\sigma\delta}^{a_1 \dots a_7} + \delta_{(\alpha}^\sigma \varepsilon_{\beta)\delta} Z_{\gamma\sigma}^{a_1 \dots a_7}, \\ [R_{\alpha\beta}, Z^{a_1 \dots a_7}] &= 0, \quad [R_{\alpha\beta}, Z^{a_1 \dots a_6, b}] = 0. \end{aligned} \quad (A.2.12)$$

The commutators with level one E_{11} generators can be taken as

$$\begin{aligned} [R_\alpha^{a_1 a_2}, P_a] &= \delta_a^{[a_1} Z_\alpha^{a_2]}, \quad [R_\alpha^{a_1 a_2}, Z_\beta^{a_3}] = -\varepsilon_{\alpha\beta} Z^{a_1 a_2 a_3}, \quad [R_\alpha^{a_1 a_2}, Z_\beta^{a_3 a_4 a_5}] = Z_\alpha^{a_1 \dots a_5}, \\ [R_\alpha^{a_1 a_2}, Z_\beta^{a_3 \dots a_7}] &= Z_{\alpha\beta}^{a_1 \dots a_7} - \varepsilon_{\alpha\beta} Z^{a_1 \dots a_7} - \varepsilon_{\alpha\beta} Z^{a_1 a_2 [a_3 \dots a_6, a_7]}. \end{aligned} \quad (A.2.13)$$

The commutators with other positive-level generators can be found using the Jacobi identities to be given by

$$\begin{aligned} [R^{a_1 \dots a_4}, P_a] &= 2 \delta_a^{[a_1} Z^{a_2 a_3 a_4]}, \quad [R^{a_1 \dots a_4}, Z_\alpha^{a_5}] = -Z_\alpha^{a_1 \dots a_5}, \\ [R^{a_1 \dots a_4}, Z^{a_5 a_6 a_7}] &= 2 Z^{a_1 \dots a_7} + \frac{3}{5} Z^{a_1 \dots a_4 [a_5 a_6, a_7]}, \quad [R_\alpha^{a_1 \dots a_6}, P_a] = \frac{3}{4} \delta_a^{[a_1} Z_\alpha^{a_2 \dots a_6]}, \end{aligned}$$

$$\begin{aligned}
[R_{\alpha}^{a_1 \dots a_6}, Z_{\beta}^{a_7}] &= -\frac{1}{4} Z_{\alpha\beta}^{a_1 \dots a_7} + \frac{3}{4} \varepsilon_{\alpha\beta} Z^{a_1 \dots a_7} + \frac{1}{20} \varepsilon_{\alpha\beta} Z^{a_1 \dots a_6, a_7}, \\
[R_{\alpha\beta}^{a_1 \dots a_8}, P_a] &= -\delta_a^{[a_1} Z_{\alpha\beta}^{a_2 \dots a_8]}, \\
[R_{a_1 \dots a_7, b}, P_a] &= -3\delta_a^b Z^{a_1 \dots a_7} + 3\delta_a^{[b} Z^{a_1 \dots a_7]} + \frac{21}{20} \delta_a^{[a_1} Z^{a_2 \dots a_7], b}. \tag{A.2.14}
\end{aligned}$$

The commutators with level -1 E_{11} generators are given by

$$\begin{aligned}
[R_{a_1 a_2}^{\alpha}, P_a] &= 0, \quad [R_{a_1 a_2}^{\alpha}, Z_{\beta}^b] = -4\delta_{\beta}^{\alpha} \delta_{[a_1}^b P_{a_2]}, \quad [R_{a_1 a_2}^{\alpha}, Z^{b_1 b_2 b_3}] = -6\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_{\beta}^{b_3]}, \\
[R_{a_1 a_2}^{\alpha}, Z_{\beta}^{b_1 \dots b_5}] &= 20\delta_{\beta}^{\alpha} \delta_{a_1 a_2}^{[b_1 b_2} Z_{\beta}^{b_3 b_4 b_5]}, \quad [R_{a_1 a_2}^{\alpha}, Z_{\alpha_1 \alpha_2}^{b_1 \dots b_7}] = 42\delta_{(\alpha_1}^{\alpha} \delta_{a_1 a_2}^{[b_1 b_2} Z_{\alpha_2]}^{b_3 \dots b_7]}, \tag{A.2.15} \\
[R_{a_1 a_2}^{\alpha}, Z^{b_1 \dots b_7}] &= -3\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_{\beta}^{b_3 \dots b_7]}, \\
[R_{a_1 a_2}^{\alpha}, Z^{b_1 \dots b_6, b}] &= -150\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_{\beta}^{b_3 \dots b_6]b} + 150\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_{\beta}^{b_3 \dots b_6 b]},
\end{aligned}$$

while the commutators with level -2 generators are

$$\begin{aligned}
[R_{a_1 \dots a_4}, P_a] &= 0, \quad [R_{a_1 \dots a_4}, Z_{\beta}^b] = 0, \quad [R_{a_1 \dots a_4}, Z^{b_1 b_2 b_3}] = 48\delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} P_{b_4]}, \\
[R_{a_1 \dots a_4}, Z_{\alpha}^{b_1 \dots b_5}] &= 120\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z_{\alpha}^{b_5]}, \quad [R_{a_1 \dots a_4}, Z_{\alpha_1 \alpha_2}^{b_1 \dots b_7}] = 0, \\
[R_{a_1 \dots a_4}, Z^{b_1 \dots b_7}] &= -120\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6 b_7]}, \\
[R_{a_1 \dots a_4}, Z^{b_1 \dots b_6, b}] &= -1800\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6]b} + 1800\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6 b]}, \tag{A.2.16}
\end{aligned}$$

with level -3 generators

$$\begin{aligned}
[R_{a_1 \dots a_6}^{\alpha}, P_a] &= 0, \quad [R_{a_1 \dots a_6}^{\alpha}, Z_{\beta}^b] = 0, \quad [R_{a_1 \dots a_6}^{\alpha}, Z^{b_1 b_2 b_3}] = 0, \\
[R_{a_1 \dots a_6}^{\alpha}, Z_{\beta}^{b_1 \dots b_5}] &= -360\delta_{\beta}^{\alpha} \delta_{[a_1 \dots a_5}^{b_1 \dots b_5} P_{a_6]}, \quad [R_{a_1 \dots a_6}^{\alpha}, Z_{\alpha_1 \alpha_2}^{b_1 \dots b_7}] = -1260\delta_{(\alpha_1}^{\alpha} \delta_{a_1 \dots a_6}^{[b_1 \dots b_6} Z_{\alpha_2]}^{b_7]}, \\
[R_{a_1 \dots a_6}^{\alpha}, Z^{b_1 \dots b_7}] &= 270\varepsilon^{\alpha\beta} \delta_{a_1 \dots a_6}^{[b_1 \dots b_6} Z_{\beta}^{b_7]}, \\
[R_{a_1 \dots a_6}^{\alpha}, Z^{b_1 \dots b_6, b}] &= 900\varepsilon^{\alpha\beta} \delta_{a_1 \dots a_6}^{b_1 \dots b_6} Z_{\beta}^b - 900\varepsilon^{\alpha\beta} \delta_{a_1 \dots a_6}^{[b_1 \dots b_6} Z_{\beta}^b], \tag{A.2.17}
\end{aligned}$$

and, finally, with level -4 generators

$$\begin{aligned}
[R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, P_a] &= 0, \quad [R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, Z_{\beta}^b] = 0, \quad [R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, Z^{b_1 b_2 b_3}] = 0, \quad [R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, Z_{\beta}^{b_1 \dots b_5}] = 0, \\
[R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, Z_{\beta_1 \beta_2}^{b_1 \dots b_7}] &= -20160\delta_{\beta_1 \beta_2}^{(\alpha_1 \alpha_2)} \delta_{[a_1 \dots a_7}^{b_1 \dots b_7} P_{a_8]}, \quad [R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, Z^{b_1 \dots b_7}] = 0, \\
[R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, Z^{b_1 \dots b_6, b}] &= 0, \quad [R_{a_1 \dots a_7, a}, P_a] = 0, \quad [R_{a_1 \dots a_7, a}, Z_{\beta}^b] = 0, \\
[R_{a_1 \dots a_7, a}, Z^{b_1 b_2 b_3}] &= 0, \quad [R_{a_1 \dots a_7, a}, Z_{\beta}^{b_1 \dots b_5}] = 0,
\end{aligned}$$

$$\begin{aligned}
[R_{a_1 \dots a_7, a}, Z_{\alpha_1 \alpha_2}^{b_1 \dots b_7}] &= 0, \quad [R_{a_1 \dots a_7, a}, Z^{b_1 \dots b_7}] = 4320 \delta_{a_1 \dots a_7}^{b_1 \dots b_7} P_a - 4320 \delta_{[a_1 \dots a_7}^{b_1 \dots b_7} P_{a]}, \\
[R_{a_1 \dots a_7, a}, Z^{b_1 \dots b_6, b}] &= -75600 \delta_a^b \delta_{[a_1 \dots a_6}^{b_1 \dots b_6} P_{a_7]} + 75600 \delta_{a[a_1 \dots a_6}^{bb_1 \dots b_6} P_{a_7]}. \tag{A.2.18}
\end{aligned}$$

A.3 $D = 5$ algebra

The E_{11} algebra for the generators decomposed into representations of $GL(5) \otimes E_6$ is given below. This algebra for $SL(5)$ and the form generators, up to level four, can be found in references [15,28] and [10], which also include some useful identities. Here we compute the full E_{11} algebra up to level four and its commutators with the l_1 representation up to level 3. These include, in particular, the generators associated with the dual graviton and were given in equation (3.3.1) and (3.3.2). By construction the generators of E_{11} are in representations of $SL(5)$ and this determined their commutators with K^a_b to be given by

$$\begin{aligned}
[K^a_b, K^c_d] &= \delta_b^c K^a_d - \delta_d^a K^c_b, \quad [K^a_b, R^\alpha] = 0, \\
[K^a_b, R^{cN}] &= \delta_b^c R^{aN}, \quad [K^a_b, R_{cN}] = -\delta_c^a R_{bN}, \\
[K^a_b, R^{a_1 a_2 N}] &= 2 \delta_b^{[a_1} R^{a|a_2]}_{N}, \quad [K^a_b, R_{a_1 a_2}^N] = -2 \delta_{[a_1}^a R_{b|a_2]}^N, \\
[K^a_b, R^{a_1 a_2 a_3, \alpha}] &= 3 \delta_b^{[a_1} R^{a|a_2 a_3], \alpha}, \quad [K^a_b, R_{a_1 a_2 a_3}^\alpha] = -3 \delta_{[a_1}^a R_{b|a_2 a_3]}^\alpha, \tag{A.3.1} \\
[K^a_b, R^{a_1 a_2, c}] &= 2 \delta_b^{[a_1} R^{a|a_2], c} + \delta_b^c R^{a_1 a_2, a}, \quad [K^a_b, R_{a_1 a_2, c}] = -2 \delta_{[a_1}^a R_{b|a_2], c} - \delta_c^a R_{a_1 a_2, b}, \\
[K^a_b, R^{a_1 \dots a_4 N_1 N_2}] &= 4 \delta_b^{[a_1} R^{a|a_2 a_3 a_4]}_{N_1 N_2}, \quad [K^a_b R_{a_1 \dots a_4}^{N_1 N_2}] = -4 \delta_{[a_1}^a R_{b|a_2 a_3 a_4]}^{N_1 N_2}, \\
[K^a_b, R^{a_1 a_2 a_3, cN}] &= 3 \delta_b^{[a_1} R^{a|a_2 a_3], cN} + \delta_b^c R^{a_1 a_2 a_3, aN}, \\
[K^a_b, R_{a_1 a_2 a_3, cN}] &= -3 \delta_{[a_1}^a R_{b|a_2 a_3], cN} - \delta_c^a R_{a_1 a_2 a_3, bN}.
\end{aligned}$$

The commutation relation of any generator with R^α is determined by the representation of E_6 that this generator belongs to:

$$\begin{aligned}
[R^\alpha, R^\beta] &= f^{\alpha\beta}{}_\gamma R^\gamma, \quad [R^\alpha, R^{aM}] = (D^\alpha)_N{}^M R^{aN}, \quad [R^\alpha, R_{aM}] = - (D^\alpha)_M{}^N R_{aN}, \\
[R^\alpha, R^{a_1 a_2 M}] &= - (D^\alpha)_M{}^N R^{a_1 a_2 N}, \quad [R^\alpha, R_{a_1 a_2}^N] = (D^\alpha)_M{}^N R_{a_1 a_2}^M, \\
[R^\alpha, R^{a_1 a_2 a_3, \beta}] &= f^{\alpha\beta}{}_\gamma R^{a_1 a_2 a_3, \gamma}, \quad [R^\alpha, R_{a_1 a_2 a_3}^\beta] = f^{\alpha\beta}{}_\gamma R_{a_1 a_2 a_3}^\gamma, \\
[R^\alpha, R^{a_1 a_2, b}] &= 0, \quad [R^\alpha, R_{a_1 a_2, b}] = 0, \\
[R^\alpha, R^{abcd}{}_{MN}] &= - (D^\alpha)_M{}^P R^{abcd}{}_{PN} - (D^\alpha)_N{}^P R^{abcd}{}_{MP}, \\
[R^\alpha, R_{abcd}{}^{MN}] &= (D^\alpha)_P{}^M R^{abcd}{}^{PN} + (D^\alpha)_P{}^N R_{abcd}{}^{MP}, \\
[R^\alpha, R^{a_1 a_2 a_3, bM}] &= (D^\alpha)_N{}^M R^{a_1 a_2 a_3, bN}, \quad [R^\alpha, R_{a_1 a_2 a_3, bM}] = - (D^\alpha)_M{}^N R_{a_1 a_2 a_3, bN}. \tag{A.3.2}
\end{aligned}$$

where $f^{\alpha\beta}{}_\gamma$ are the structure constants of E_6 , normalised by $f_{\alpha\beta\gamma} f^{\alpha\beta\delta} = -4 \delta_\gamma^\delta$, $(D^\alpha)_N{}^M$ are the generators of E_6 in **27** representation. We lower and raise indices with the Killing metric $g_{\alpha\beta}$.

The commutation relations of the positive level E_{11} generators are given by

$$\begin{aligned}
[R^{aM}, R^{bN}] &= d^{MNP} R^{ab}{}_P, \quad [R^{aN}, R^{bc}{}_M] = (D_\alpha)_M{}^N R^{abc, \alpha} + \delta_M^N R^{bc, a}, \\
[R^{ab}{}_M, R^{cd}{}_N] &= R^{abcd}{}_{MN} - 20 d_{MNP} R^{ab[c, d]P}, \quad [R^{aN}, R^{bc, d}] = R^{abc, dN} - \frac{1}{3} R^{bcd, aN}, \\
[R^{aN}, R^{bcd, \alpha}] &= 3 d^{NMP} (D^\alpha)_P{}^R R^{abcd}{}_{MR} + 6 (D^\alpha)_M{}^N R^{bcd, aM}. \tag{A.3.3}
\end{aligned}$$

where d^{MNP} is the completely symmetric invariant tensor of E_6 , normalised by $d_{NPR} d^{MPR} = \delta_N^M$.

The commutators of negative-level E_{11} generators are given by

$$\begin{aligned}
[R_{aN}, R_{bM}] &= d_{NMP} R_{ab}{}^P, \quad [R_{aN}, R_{bc}{}^M] = (D_\alpha)_N{}^M R_{abc}{}^\alpha + \delta_N^M R_{bc, a}, \\
[R_{ab}{}^M, R_{cd}{}^N] &= R_{abcd}{}^{MN} - 20 d^{MNP} R_{ab[c, d]P}, \quad [R_{aN}, R_{bc, d}] = R_{abc, dN} - \frac{1}{3} R_{bcd, aN}, \\
[R_{aN}, R_{bcd, \alpha}] &= 3 d_{NMP} (D_\alpha)_R{}^P R_{abcd}{}^{MR} + 6 (D_\alpha)_N{}^M R_{bcd, aM}. \tag{A.3.4}
\end{aligned}$$

The commutators between the positive and negative level generators of E_{11} up to level 4 are given by

$$\begin{aligned}
[R^{aN}, R_{bM}] &= 6 \delta_a^b (D_\alpha)_M{}^N R^\alpha + \delta_M^N K^a{}_b - \frac{1}{3} \delta_M^N \delta_b^a K^c{}_c, \\
[R_{aN}, R^{bc}{}_M] &= 20 d_{NMP} \delta_a^{[b} R^{c]P}, \quad [R^{aN}, R_{bc}{}^M] = 20 d^{NMP} \delta_{[b}^a R_{c]P}, \\
[R_{aN}, R^{a_1 a_2 a_3, \alpha}] &= 18 (D^\alpha)_N{}^M \delta_a^{[a_1} R^{a_2 a_3]}{}_M, \quad [R^{aN}, R_{a_1 a_2 a_3}{}^\alpha] = 18 (D^\alpha)_M{}^N \delta_{[a_1}^a R_{a_2 a_3]}^M, \\
[R_{aN}, R^{a_1 a_2, b}] &= \delta_a^b R^{a_1 a_2}{}_N - \delta_a^{[b} R^{a_1 a_2]}{}_N, \quad [R^{aN}, R_{a_1 a_2, b}] = \delta_b^a R_{a_1 a_2}{}^N - \delta_{[b}^a R_{a_1 a_2]}^N, \\
[R_{aN}, R^{a_1 \dots a_4}{}_{N_1 N_2}] &= 40 d_{N[N_1|M|} (D_\alpha)_{N_2]}{}^M \delta_a^{[a_1} R^{a_2 a_3 a_4]}{}_M, \\
[R^{aN}, R_{a_1 \dots a_4}{}^{N_1 N_2}] &= 40 d^{N[N_1|M|} (D_\alpha)_{M}{}^{N_2]} \delta_{[a_1}^a R_{a_2 a_3 a_4]}^N, \\
[R_{aN}, R^{a_1 a_2 a_3, bM}] &= (D_\alpha)_N{}^M \delta_a^b R^{a_1 a_2 a_3, \alpha} - (D_\alpha)_N{}^M \delta_a^{[b} R^{a_1 a_2 a_3]}{}_M, \alpha + 3 \delta_N^M \delta_{[a_1}^a R^{a_2 a_3]}{}_M, b, \\
[R^{aN}, R_{a_1 a_2 a_3, bM}] &= (D_\alpha)_M{}^N \delta_b^a R_{a_1 a_2 a_3}{}^\alpha - (D_\alpha)_M{}^N \delta_{[b}^a R_{a_1 a_2 a_3]}^N, \alpha + 3 \delta_M^N \delta_{[a_1}^a R_{a_2 a_3]}^N, b. \tag{A.3.5}
\end{aligned}$$

The Cartan involution acts on the generators of E_{11} as follows

$$\begin{aligned}
I_c(K^a{}_b) &= -K^b{}_a, \quad I_c(R^\alpha) = -R^{-\alpha}, \quad I_c(R^{aN}) = -J^{MN} R_{aM}, \\
I_c(R^{ab}{}_M) &= J_{MN}^{-1} R_{ab}{}^N, \quad I_c(R^{abc, \alpha}) = -R_{abc, -\alpha}, \quad I_c(R^{a_1 a_2, c}) = -R_{a_1 a_2, c}, \\
I_c(R^{abcd}{}_{MN}) &= J_{MP}^{-1} J_{NQ}^{-1} R_{abcd}{}^{PQ}, \quad I_c(R^{abc, dN}) = J^{NM} R_{abc, dM}. \tag{A.3.6}
\end{aligned}$$

We now give the commutators between the generators of E_{11} and those of the l_1 representation given in (3.3.3) up to level 3. The commutation relations between the later and the generators of $GL(5)$ are given by

$$\begin{aligned} [K^a{}_b, P_c] &= -\delta_c^a P_b + \frac{1}{2} \delta_b^a P_c, \quad [K^a{}_b, Z^N] = \frac{1}{2} \delta_b^a Z^N, \quad [K^a{}_b, Z^c{}_N] = \delta_b^c Z_N^a + \frac{1}{2} \delta_b^a Z_N^c, \\ [K^a{}_b, Z^{a_1 a_2, \alpha}] &= 2 \delta_b^{[a_1} Z^{a|a_2], \alpha} + \frac{1}{2} \delta_b^a Z^{a_1 a_2, \alpha}, \quad [K^a{}_b, Z^{cd}] = \delta_b^c Z^{ad} + \delta_b^d Z^{ca} + \frac{1}{2} \delta_b^a Z^{cd}. \end{aligned} \quad (A.3.7)$$

while with the generators of E_6 we have

$$\begin{aligned} [R^\alpha, P_a] &= 0, \quad [R^\alpha, Z^M] = (D^\alpha)_N{}^M Z^N, \quad [R^\alpha, Z^a{}_N] = - (D^\alpha)_N{}^M Z^a{}_M, \\ [R^\alpha, Z^{a_1 a_2, \beta}] &= f^{\alpha\beta}{}_\gamma Z^{a_1 a_2, \gamma}, \quad [R^\alpha, Z^{ab}] = 0. \end{aligned} \quad (A.3.8)$$

The elements of the l_1 representation at a given level can be introduced into the algebra by taking the commutators of suitable E_{11} generators of the same level with P_a , namely

$$\begin{aligned} [R^{aN}, P_b] &= \delta_b^a Z^N, \quad [R^{a_1 a_2}{}_N, P_a] = 2 \delta_a^{[a_1} Z^{a_2]}{}_N, \\ [R^{a_1 a_2 a_3, \alpha}, P_a] &= 3 \delta_a^{[a_1} Z^{a_2 a_3], \alpha}, \quad [R^{a_1 a_2, b}, P_a] = -2 \delta_a^b Z^{[a_1 a_2]} - 2 \delta_a^{[a_1} Z^{b|a_2]}, \end{aligned}$$

The commutators of the remaining positive level generators of E_{11} with the l_1 generators is determined by the Jacobi identities and they are found to be given by

$$\begin{aligned} [R^{aM}, Z^N] &= -d^{MNP} Z^a{}_P, \quad [R^{aN}, Z^b{}_M] = - (D_\alpha)_M{}^N Z^{ab, \alpha} - \delta_M^N Z^{ab}, \\ [R^{a_1 a_2}{}_N, Z^M] &= - (D_\alpha)_N{}^M Z^{a_1 a_2, \alpha} + 2 \delta_M^N Z^{[a_1 a_2]}. \end{aligned} \quad (A.3.9)$$

Commutators between the level -1 generators of E_{11} and those of the l_1 representation are also determined by the Jacobi identities to be given by

$$\begin{aligned} [R_{aN}, P_b] &= 0, \quad [R_{aN}, Z^M] = \delta_N^M P_a, \quad [R_{aN}, Z^b{}_M] = -10 d_{NMP} \delta_a^b Z^P, \\ [R_{aN}, P_b] &= 0, \quad [R_{aN}, Z^M] = \delta_N^M P_a, \quad [R_{aN}, Z^b{}_M] = -10 d_{NMP} \delta_a^b Z^P, \\ [R_{aN}, Z^{a_1 a_2, \alpha}] &= -12 (D^\alpha)_N{}^M \delta_a^{[a_1} Z^{a_2]}{}_M, \quad [R_{aN}, Z^{bc}] = -\frac{2}{3} \delta_a^b Z^c{}_N - \frac{1}{3} \delta_a^c Z^b{}_N. \end{aligned} \quad (A.3.10)$$

A.4 $D = 4$ algebra

In this appendix we give the $E_{11} \otimes_s l_1$ algebra decomposed into representations of $GL(4) \times SL(8)$ that corresponds to four-dimensional theory [19]. This latter reference contains a few typographical errors in the commutators which are corrected here. We will

first give the commutation relations of level 0 generators with the rest of E_{11} algebra. The commutation relations of any generator with K^a_b are

$$\begin{aligned}
[K^a_b, K^c_d] &= \delta_b^c K^a_d - \delta_d^a K^c_b, \quad [K^a_b, R^I_J] = 0, \quad [K^a_b, R^{I_1 \dots I_4}] = 0, \\
[K^a_b, R^{c I_1 I_2}] &= \delta_b^c R^{a I_1 I_2}, \quad [K^a_b, R^c_{I_1 I_2}] = \delta_b^c R^a_{I_1 I_2}, \\
[K^a_b, \tilde{R}_{c I_1 I_2}] &= -\delta_c^a \tilde{R}_{b I_1 I_2}, \quad [K^a_b, \tilde{R}_c^{I_1 I_2}] = -\delta_c^a \tilde{R}_b^{I_1 I_2}, \\
[K^a_b, \hat{K}^{cd}] &= 2\delta_b^{(c} \hat{K}^{a|d)}, \quad [K^a_b, \hat{\tilde{K}}_{cd}] = -2\delta_{(c}^a \hat{\tilde{K}}_{b|d)}, \\
[K^a_b, R^{a_1 a_2 I}_J] &= 2\delta_b^{[a_1} R^{a|a_2]I}_J, \quad [K^a_b, \tilde{R}_{a_1 a_2}^I_J] = -2\delta_{[a_1}^a \tilde{R}_{b|a_2]}^I_J, \\
[K^a_b, R^{a_1 a_2 I_1 \dots I_4}] &= 2\delta_b^{[a_1} R^{a|a_2]I_1 \dots I_4}, \quad [K^a_b, \tilde{R}_{a_1 a_2 I_1 \dots I_4}] = -2\delta_{[a_1}^a \tilde{R}_{b|a_2]I_1 \dots I_4}.
\end{aligned} \tag{A.4.1}$$

The commutators with $SL(8)$ generator R^I_J are given by

$$\begin{aligned}
[R^I_J, R^K_L] &= \delta_J^K R^I_L - \delta_L^I R^K_J, \quad [R^I_J, R^{I_1 \dots I_4}] = 4\delta_J^{[I_1} R^{I|I_2 I_3 I_4]} - \frac{1}{2}\delta_J^I R^{I_1 \dots I_4}, \\
[R^I_J, R^{a I_1 I_2}] &= 2\delta_J^{[I_1} R^{a|I|I_2]} - \frac{1}{4}\delta_J^I R^{a I_1 I_2}, \quad [R^I_J, R^a_{I_1 I_2}] = -2\delta_{[I_1}^I R^a_{J|I_2]} + \frac{1}{4}\delta_J^I R^a_{I_1 I_2}, \\
[R^I_J, \tilde{R}_{a I_1 I_2}] &= -2\delta_{[I_1}^I \tilde{R}_{a|J|I_2]} + \frac{1}{4}\delta_J^I \tilde{R}_{a I_1 I_2}, \quad [R^I_J, \tilde{R}_a^{I_1 I_2}] = 2\delta_J^{[I_1} \tilde{R}_a^{I|I_2]} - \frac{1}{4}\delta_J^I \tilde{R}_a^{I_1 I_2}, \\
[R^I_J, \hat{K}^{(ab)}] &= 0, \quad [R^I_J, \hat{\tilde{K}}_{(ab)}] = 0, \\
[R^I_J, R^{a_1 a_2 K}_L] &= \delta_J^K R^{a_1 a_2 I}_L - \delta_L^I R^{a_1 a_2 K}_J, \\
[R^I_J, \tilde{R}_{a_1 a_2}^K_L] &= \delta_J^K \tilde{R}_{a_1 a_2 K}^I_L - \delta_L^I \tilde{R}_{a_1 a_2}^K_J, \\
[R^I_J, R^{a_1 a_2 I_1 \dots I_4}] &= 4\delta_J^{[I_1} R^{a_1 a_2|I|I_2 I_3 I_4]} - \frac{1}{2}\delta_J^I R^{a_1 a_2 I_1 \dots I_4}, \\
[R^I_J, \tilde{R}_{a_1 a_2 I_1 \dots I_4}] &= -4\delta_{[I_1}^I \tilde{R}_{a_1 a_2|J|I_2 I_3 I_4]} + \frac{1}{2}\delta_J^I \tilde{R}_{a_1 a_2 I_1 \dots I_4},
\end{aligned} \tag{A.4.2}$$

The commutators with the other E_7 generators $R^{I_1 \dots I_4}$ generators are given by

$$\begin{aligned}
[R^{I_1 \dots I_4}, R^{J_1 \dots J_4}] &= -\frac{1}{36}\varepsilon^{I_1 \dots I_4 [J_1 J_2 J_3 | L |} R^{J_4]}_L, \\
[R^{I_1 \dots I_4}, R^{a J_1 J_2}] &= \frac{1}{24}\varepsilon^{I_1 \dots I_4 J_1 \dots J_4} R^a_{J_3 J_4}, \quad [R^{I_1 \dots I_4}, R^a_{J_1 J_2}] = \delta_{J_1 J_2}^{[I_1 I_2} R^{a I_3 I_4]}, \\
[R^{I_1 \dots I_4}, \tilde{R}_{a J_1 J_2}] &= \delta_{J_1 J_2}^{[I_1 I_2} \tilde{R}_a^{I_3 I_4]}, \quad [R^{I_1 \dots I_4}, \tilde{R}_a^{J_1 J_2}] = \frac{1}{24}\varepsilon^{I_1 \dots I_4 J_1 \dots J_4} \tilde{R}_{a J_3 J_4},
\end{aligned}$$

$$\begin{aligned}
[R^{I_1 \dots I_4}, R^{a_1 a_2 I}_J] &= -4 \delta_J^{[I_1} R^{a_1 a_2 | I | I_2 I_3 I_4]} + \frac{1}{2} \delta_J^I R^{a_1 a_2 I_1 \dots I_4}, \\
[R^{I_1 \dots I_4}, \tilde{R}_{a_1 a_2}{}^I{}_J] &= -\frac{1}{6} \varepsilon^{I_1 \dots I_4 J_1 J_2 J_3 I} \tilde{R}_{a_1 a_2 J_1 J_2 J_3 J} + \frac{1}{48} \delta_J^I \varepsilon^{I_1 \dots I_4 J_1 \dots J_4} \tilde{R}_{a_1 a_2 J_1 \dots J_4}, \\
[R^{I_1 \dots I_4}, \hat{K}^{(ab)}] &= 0, \quad [R^{I_1 \dots I_4}, \hat{K}_{(ab)}] = 0, \\
[R^{I_1 \dots I_4}, R^{a_1 a_2 J_1 \dots J_4}] &= \frac{1}{36} \varepsilon^{I_1 \dots I_4 [J_1 J_2 J_3 | L |} R^{a_1 a_2 J_4]}{}_L, \\
[R^{I_1 \dots I_4}, \tilde{R}_{a_1 a_2 J_1 \dots J_4}] &= -\frac{2}{3} \delta_{[J_1 J_2 J_3}^{[I_1 I_2 I_3} \tilde{R}_{a_1 a_2}{}^{I_4]}{}_{J_4]}. \tag{A.4.3}
\end{aligned}$$

The commutators of the positive level one E_{11} generators with each other are given by

$$\begin{aligned}
[R^{a I_1 I_2}, R^{b I_3 I_4}] &= -12 R^{ab I_1 \dots I_4}, \quad [R^a{}_{I_1 I_2}, R^b{}_{I_3 I_4}] = \frac{1}{2} \varepsilon_{I_1 \dots I_4 J_1 \dots J_4} R^{ab J_1 \dots J_4}, \\
[R^{a I_1 I_2}, R^b{}_{J_1 J_2}] &= 4 \delta_{[J_1}^{[I_1} R^{ab I_2]}{}_{J_2]} + 2 \delta_{J_1 J_2}^{I_1 I_2} \hat{K}^{(ab)}. \tag{A.4.4}
\end{aligned}$$

The equivalent commutators for the negative level E_{11} generators are

$$\begin{aligned}
[\tilde{R}_{a I_1 I_2}, \tilde{R}_{b I_3 I_4}] &= -12 \tilde{R}_{ab I_1 \dots I_4}, \quad [\tilde{R}_a{}^{I_1 I_2}, \tilde{R}_b{}^{I_3 I_4}] = \frac{1}{2} \varepsilon^{I_1 \dots I_4 J_1 \dots J_4} \tilde{R}_{a_1 a_2 J_1 \dots J_4}, \\
[\tilde{R}_{a I_1 I_2}, \tilde{R}_b{}^{J_1 J_2}] &= 4 \delta_{[I_1}^{[J_1} \tilde{R}_{ab}{}^{J_2]}{}_{I_2]} + 2 \delta_{I_1 I_2}^{J_1 J_2} \hat{K}_{(ab)}. \tag{A.4.5}
\end{aligned}$$

The commutators between the level 1 and -1 E_{11} generators are given by

$$\begin{aligned}
[R^{a I_1 I_2}, \tilde{R}_{b J_1 J_2}] &= 2 \delta_{J_1 J_2}^{I_1 I_2} K^a{}_b + 4 \delta_b^a \delta_{[J_1}^{[I_1} K^{I_2]}{}_{J_2]} - \delta_b^a \delta_{J_1 J_2}^{I_1 I_2} K^c{}_c, \\
[R^a{}_{I_1 I_2}, \tilde{R}_b{}^{J_1 J_2}] &= -2 \delta_{I_1 I_2}^{J_1 J_2} K^a{}_b + 4 \delta_b^a \delta_{[I_1}^{[J_1} K^{J_2]}{}_{I_2]} + \delta_b^a \delta_{I_1 I_2}^{J_1 J_2} K^c{}_c, \\
[R^{a I_1 I_2}, \tilde{R}_b{}^{I_3 I_4}] &= -12 \delta_b^a R^{I_1 \dots I_4}, \quad [R^a{}_{I_1 I_2}, \tilde{R}_{b I_3 I_4}] = \frac{1}{2} \delta_b^a \varepsilon_{I_1 \dots I_4 J_1 \dots J_4} R^{J_1 \dots J_4}. \tag{A.4.6}
\end{aligned}$$

The commutators with the level 2 and level -1 E_{11} generators are given by

$$\begin{aligned}
[R^{ab I}_J, \tilde{R}_{c I_1 I_2}] &= -4 \delta_c^{[a} \delta_{[I_1}^I R^{b]}{}_{J | I_2]} + \frac{1}{2} \delta_c^{[a} \delta_J^I R^{b]}{}_{I_1 I_2}, \\
[R^{ab I}_J, \tilde{R}_c{}^{I_1 I_2}] &= 4 \delta_c^{[a} \delta_J^{[I_1} R^{b]}{}_{| I | I_2]} - \frac{1}{2} \delta_c^{[a} \delta_J^I R^{b]}{}^{I_1 I_2}, \\
[R^{ab I_1 \dots I_4}, \tilde{R}_{c J_1 J_2}] &= 2 \delta_c^{[a} \delta_{J_1 J_2}^{[I_1 I_2} R^{b]}{}_{I_3 I_4]}, \quad [R^{ab I_1 \dots I_4}, \tilde{R}_c{}^{I_5 I_6}] = \frac{1}{12} \varepsilon^{I_1 \dots I_8} \delta_c^{[a} R^{b]}{}_{I_7 I_8}, \\
[\hat{K}^{ab}, \tilde{R}_{c I_1 I_2}] &= -\delta_c^{(a} R^{b)}{}_{I_1 I_2}, \quad [\hat{K}^{ab}, \tilde{R}_c{}^{I_1 I_2}] = -\delta_c^{(a} R^{b)}{}^{J_1 J_2}. \tag{A.4.7}
\end{aligned}$$

Finally, the commutators of level -2 with the level 1 E_{11} generators are

$$\begin{aligned}
\left[\tilde{R}_{ab}{}^I{}_J, R^{cI_1 I_2} \right] &= -4 \delta_{[a}^c \delta_J^{[I_1} \tilde{R}_{b]}^{I_2]} + \frac{1}{2} \delta_{[a}^c \delta_J^I \tilde{R}_{b]}^{I_1 I_2}, \\
\left[\tilde{R}_{ab}{}^I{}_J, R^c{}_{I_1 I_2} \right] &= 4 \delta_{[a}^c \delta_{[I_1}^I \tilde{R}_{b]}^{I_2]} - \frac{1}{2} \delta_{[a}^c \delta_J^I \tilde{R}_{b]}^{I_1 I_2}, \\
\left[\tilde{R}_{ab}{}^{I_1 \dots I_4}, R^c{}_{J_1 J_2} \right] &= 2 \delta_{[a}^c \delta_{J_1 J_2}^{[I_1 I_2} \tilde{R}_{b]}^{I_3 I_4]}, \quad \left[\tilde{R}_{ab}{}^{I_1 \dots I_4}, R^{cI_5 I_6} \right] = \frac{1}{12} \varepsilon^{I_1 \dots I_8} \delta_{[a}^c \tilde{R}_{b]}^{I_7 I_8}, \\
\left[\hat{\tilde{K}}_{ab}, R^c{}_{I_1 I_2} \right] &= -\delta_{(a}^c \tilde{R}_{b) I_1 I_2}, \quad \left[\hat{\tilde{K}}_{ab}, R^{cI_1 I_2} \right] = -\delta_{(a}^c \tilde{R}_{b)}^{I_1 I_2}. \tag{A.4.8}
\end{aligned}$$

The Cartan involution preserves the above commutators and is given by

$$\begin{aligned}
I_c(K^a{}_b) &= -K^b{}_a, \quad I_c(R^I{}_J) = -R^J{}_I, \quad I_c(R^{I_1 \dots I_4}) = -\star R^{I_1 \dots I_4} \equiv -\frac{1}{4!} \varepsilon^{I_1 \dots I_4 J_1 \dots J_4} R^{J_1 \dots J_4}, \\
I_c(R^{aI_1 I_2}) &= -\tilde{R}_{aI_1 I_2}, \quad I_c(R^a{}_{I_1 I_2}) = \tilde{R}_a{}^{I_1 I_2} \\
I_c(R^{a_1 a_2 I}{}_J) &= -\tilde{R}_{a_1 a_2}{}^J{}_I, \quad I_c(R^{a_1 a_2 I_1 \dots I_4}) = \tilde{R}_{a_1 a_2 I_1 \dots I_4}, \quad I_c(\hat{K}^{ab}) = -\tilde{\hat{K}}^{ab}
\end{aligned}$$

We now give the action of E_{11} on the l_1 representation generators whose elements were given in equation (3.4.3). The commutation relations of the l_1 representation with level 0 generators of E_{11} are given by

$$\begin{aligned}
[K^a{}_b, P_c] &= -\delta_c^a P_b + \frac{1}{2} \delta_b^a P_c, \quad [K^a{}_b, Z^{I_1 I_2}] = \frac{1}{2} \delta_b^a Z^{I_1 I_2}, \quad [K^a{}_b, Z_{I_1 I_2}] = \frac{1}{2} \delta_b^a Z_{I_1 I_2}, \\
[K^a{}_b, Z^c] &= \delta_b^c Z^a + \frac{1}{2} \delta_b^a Z^c, \quad [K^a{}_b, Z^{cI}{}_J] = \delta_b^c Z^a{}^I{}_J + \frac{1}{2} \delta_b^a Z^{cI}{}_J, \\
[K^a{}_b, Z^{cI_1 \dots I_4}] &= \delta_b^c Z^a{}^{I_1 \dots I_4} + \frac{1}{2} \delta_b^a Z^{cI_1 \dots I_4}, \\
[R^I{}_J, P_c] &= 0, \quad [R^I{}_J, Z^{I_1 I_2}] = 2 \delta_J^{[I_1} Z^{I_2]} - \frac{1}{4} \delta_J^I Z^{I_1 I_2}, \\
[R^I{}_J, Z_{I_1 I_2}] &= -2 \delta_{[I_1}^I Z_{J I_2]} + \frac{1}{4} \delta_J^I Z_{I_1 I_2}, \\
[R^I{}_J, Z^a] &= 0, \quad [R^I{}_J, Z^{aK}{}_L] = \delta_J^K Z^a{}^I{}_L - \delta_L^I Z^{aK}{}_J, \\
[R^I{}_J, Z^{aI_1 \dots I_4}] &= 4 \delta_J^{[I_1} Z^{aI_2 \dots I_4]} - \frac{1}{2} \delta_J^I Z^{aI_1 \dots I_4}, \\
[R^{I_1 \dots I_4}, P_a] &= 0, \quad [R^{I_1 \dots I_4}, Z^{J_1 J_2}] = \frac{1}{24} \varepsilon^{I_1 \dots I_4 J_1 \dots J_4} Z_{J_3 J_4}, \\
[R^{I_1 \dots I_4}, Z_{J_1 J_2}] &= \delta_{J_1 J_2}^{[I_1 I_2} Z^{I_3 I_4]},
\end{aligned}$$

$$\begin{aligned}
[R^{I_1 \dots I_4}, Z^a] &= 0, \quad [R^{I_1 \dots I_4}, Z^{aI}{}_J] = -\frac{4}{3} \delta_J^{[I_1} Z^{a|I|I_2 I_3 I_4]} + \frac{1}{6} \delta_J^I Z^{aI_1 \dots I_4}, \\
[R^{I_1 \dots I_4}, Z^{aJ_1 \dots J_4}] &= \frac{1}{12} \varepsilon^{J_1 \dots J_4 [I_1 I_2 I_3 | K|} Z^{aI_4]}{}_K.
\end{aligned} \tag{A.4.9}$$

The commutators with the E_{11} level 1 generators are given by

$$\begin{aligned}
[R^{aI_1 I_2}, P_b] &= \delta_b^a Z^{I_1 I_2}, \quad [R_{I_1 I_2}^a, P_b] = \delta_b^a Z_{I_1 I_2}, \\
[R^{aI_1 I_2}, Z^{J_1 J_2}] &= -Z^{aI_1 I_2 J_1 J_2}, \quad [R_{I_1 I_2}^a, Z^{J_1 J_2}] = \delta_{[I_1}^{J_1} Z^{aJ_2]}_{I_2] + \delta_{I_1 I_2}^{J_1 J_2} Z^a, \\
[R^{aI_1 I_2}, Z_{J_1 J_2}] &= \delta_{[J_1}^{[I_1} Z^{aI_2]}_{J_2]} - \delta_{J_1 J_2}^{I_1 I_2} Z^a, \quad [R_{I_1 I_2}^a, Z_{J_1 J_2}] = \frac{1}{24} \varepsilon_{I_1 I_2 J_1 J_2 K_1 \dots K_4} Z^{aK_1 \dots K_4}.
\end{aligned} \tag{A.4.10}$$

The commutators with the E_{11} level 2 generators are given by

$$\begin{aligned}
[\hat{K}^{(a_1 a_2)}, P_a] &= \delta_a^{(a_1} Z^{a_2)}, \quad [R^{a_1 a_2 I}{}_J, P_a] = -\frac{1}{2} \delta_a^{[a_1} Z^{a_2]I}{}_J, \\
[R^{a_1 a_2 I_1 \dots I_4}, P_a] &= -\frac{1}{6} \delta_a^{[a_1} Z^{a_2]I_1 \dots I_4}.
\end{aligned} \tag{A.4.11}$$

The commutators with the E_{11} level -1 generators are

$$\begin{aligned}
[\tilde{R}_{aI_1 I_2}, P_b] &= 0, \quad [\tilde{R}_{aI_1 I_2}, Z^{J_1 J_2}] = 2 \delta_{J_1 J_2}^{I_1 I_2} P_a, \quad [\tilde{R}_{aI_1 I_2}, Z_{J_1 J_2}] = 0, \\
[\tilde{R}_a^{I_1 I_2}, P_b] &= 0, \quad [\tilde{R}_a^{I_1 I_2}, Z^{J_1 J_2}] = 0, \quad [\tilde{R}_a^{I_1 I_2}, Z_{J_1 J_2}] = -2 \delta_{I_1 I_2}^{J_1 J_2} P_a, \\
[\tilde{R}_{aI_1 I_2}, Z^b] &= -2 \delta_a^b Z_{I_1 I_2}, \quad [\tilde{R}_{aI_1 I_2}, Z^{bI}{}_J] = -8 \delta_a^b \delta_{[I_1}^I Z_{I_2]J}, \\
[\tilde{R}_{aI_1 I_2}, Z^{bJ_1 \dots J_4}] &= -12 \delta_a^b \delta_{I_1 I_2}^{[J_1 J_2} Z^{J_3 J_4]}, \quad [\tilde{R}_a^{I_1 I_2}, Z^b] = -2 \delta_a^b Z^{I_1 I_2}, \\
[\tilde{R}_a^{I_1 I_2}, Z^{bI}{}_J] &= 8 \delta_a^b \delta_J^{[I_1} Z^{I_2]I}, \quad [\tilde{R}_a^{I_1 I_2}, Z^{bJ_1 \dots J_4}] = -\frac{1}{2} \delta_a^b \varepsilon^{J_1 \dots J_2 I_1 \dots I_4} Z_{I_3 I_4}.
\end{aligned} \tag{A.4.12}$$

The last three commutators in equation (A.4.9), the last four in equation (A.4.10), all in equation (A.4.11) and the last six in equation (A.4.12) are not contained in reference [19] and are taken from a forthcoming publication with Nikolay Gromov.

References

- [1] S. Weinberg, Phys. Rev. Lett. **16** (1966) 63; Phys. Rev. Lett. **18** (1967) 188; Phys. Rev. **166** (1968) 1568; Physica **96A** (1979) 327.
- [2] S. Coleman, J. Wess and B. Zumino, *Structure of Phenomenological Lagrangians. 1*, Phys. Rev. **177** (1969) 2239; C. Callan, S. Coleman, J. Wess and B. Zumino, *Structure of phenomenological Lagrangians. 2*, Phys. Rev. **177** (1969) 2247.
- [3] A. Salam and J. Strathdee, *Superfields and Fermi-Bose symmetry*, Phys. Rev. **D 11** (1975) 1521; *Feynman rules for superfields*, Nucl. Phys. **B 86** (1975) 142.

- [4] A. Borisov and V. Ogievetsky, *Theory of dynamical affine and conformal symmetries as the theory of the gravitational field*, Teor. Mat. Fiz. 21 (1974) 32.
- [5] P. West, *Hidden superconformal symmetries of M-theory*, **JHEP 0008** (2000) 007, [arXiv:hep-th/0005270](#).
- [6] P. West, *E_{11} , $SL(32)$ and Central Charges*, Phys. Lett. **B 575** (2003) 333-342, [hep-th/0307098](#)
- [7] N. Lambert and P. West, *Coset Symmetries in Dimensionally Reduced Bosonic String Theory*, Nucl.Phys. B615 (2001) 117-132, [hep-th/0107209](#).
- [8] F. Englert, L. Houart, A. Taormina and P. West, *The Symmetry of M-Theories*, JHEP 0309 (2003) 020, [hep-th/0304206](#).
- [9] P. West, *Introduction to Strings and Branes*, Cambridge University Press, June 2012.
- [10] P. West, *Generalised Space-time and Gauge Transformations*, [arXiv:1403.6395](#).
- [11] P. West, *E_{11} and M Theory*, Class. Quant. Grav. **18**(2001) 4443, [arXiv:hep-th/0104081](#).
- [12] I. Schnakenburg and P. West, *Kac-Moody symmetries of IIB supergravity*, Phys. Lett. **B517** (2001) 421, [arXiv:hep-th/0107181](#).
- [13] P. West, *The IIA, IIB and eleven dimensional theories and their common E_{11} origin*, Nucl. Phys. B693 (2004) 76-102, [hep-th/0402140](#).
- [14] F. Riccioni and P. West, *The E_{11} origin of all maximal supergravities*, JHEP **0707** (2007) 063; [arXiv:0705.0752](#).
- [15] F. Riccioni and P. West, *$E(11)$ -extended spacetime and gauged supergravities*, JHEP **0802** (2008) 039, [arXiv:0712.1795](#).
- [16] P. West, *E_{11} origin of Brane charges and U-duality multiplets*, JHEP 0408 (2004) 052, [hep-th/0406150](#).
- [17] P. West, *Brane dynamics, central charges and E_{11}* , [hep-th/0412336](#).
- [18] C. Hillmann, *Generalized $E(7(7))$ coset dynamics and $D=11$ supergravity*, JHEP **0903**, 135 (2009), [hep-th/0901.1581](#) ; *$E(7(7))$ and $d=11$ supergravity*, PhD thesis, [arXiv:0902.1509](#).
- [19] P. West, *E_{11} , Generalised space-time and equations of motion in four dimensions*, JHEP 1212 (2012) 068, [arXiv:1206.7045](#).
- [20] D. Berman, H. Godazgar, M. Perry and P. West, *Duality Invariant Actions and Generalised Geometry*, JHEP 1202 (2012) 108, [arXiv:1111.0459](#)
- [21] P. West, *Generalised Geometry, eleven dimensions and E_{11}* , JHEP 1202 (2012) 018, [arXiv:1111.1642](#).
- [22] W. Siegel, *Two vielbein formalism for string inspired axionic gravity*, Phys.Rev. D47 (1993) 5453, [hep-th/9302036](#); *Superspace duality in low-energy superstrings*, Phys.Rev. D48 (1993) 2826-2837, [hep-th/9305073](#); *Manifest duality in low-energy superstrings*, In *Berkeley 1993, Proceedings, Strings '93* 353, [hep-th/9308133](#).
- [23] P. West, *E_{11} , generalised space-time and IIA string theory*, Phys.Lett.B696 (2011) 403-409, [arXiv:1009.2624](#).
- [24] A. Rocen and P. West, *E_{11} , generalised space-time and IIA string theory; the R-R sector*, [arXiv:1012.2744](#).
- [25] D. Berman, M. J. Perry, *Generalized Geometry and M theory*, JHEP 1106 (2011) 74 [arXiv:1008.1763](#); D. Berman, H. Godazgar and M. J. Perry, *$SO(5,5)$ duality in*

- M-theory and generalized geometry*, Phys. Lett. B700 (2011) 65-67. arXiv:1103.5733.
- [26] M. Duff, *Duality Rotations In String Theory*, Nucl. Phys. B **335** (1990) 610; M. Duff and J. Lu, Duality rotations in membrane theory, Nucl. Phys. **B347** (1990) 394.
 - [27] P. West, *E_{11} , ten forms and supergravity*, JHEP 0603 (2006) 072, hep-th/0511153.
 - [28] F. Riccioni, D. Steele and P. West, *The $E(11)$ origin of all maximal supergravities - the hierarchy of field-strengths* JHEP **0909** (2009) 095, arXiv:0906.1177.
 - [29] P. West, *Generalised Space-time and Gauge Transformations*, arXiv:1403.6395.
 - [30] T. Nutma, SimPLie, a simple program for Lie algebras, <https://code.google.com/p/simplie/>.